Some New Type of Fuzzy *I*-Convergent Double Difference Sequence Spaces

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Abstract— ***In this paper we introduce some new classes of double difference sequence spaces of fuzzy numbers. We study different topological properties of these sequence spaces like completeness, solidity etc. Also we obtain some inclusion relation involving these sequence spaces.***

Keywords — Fuzzy real number, Difference sequence, Solid space, Monotone, Sequence algebra.

I. INTRODUCTION

Since the introduction of fuzzy set theory by Zadeh [30] in 1965, fuzzy logic has become an important area of research in various branches of Mathematics such as Metric and Topological spaces [4], Theory of functions [29], Approximation theory [1] etc. Fuzzy set theory also finds its applications for modeling, uncertainty and vagueness in various fields of Science and Engineering, e.g. Computer Programming [9], Nonlinear Dynamical Systems [11], Population Dynamics [2], Control of Chaos [8], Quantum Physics [16] etc. It attracted workers on sequence spaces to introduce different type of classes of sequences of fuzzy numbers.

The initial works on double sequences is found in Bromwich [3]. The notion of regular convergence of double sequences of real or complex terms is introduced by Hardy [10]. Tripathy and Dutta [26] introduced and investigated different types of fuzzy real valued double sequence spaces. Generalizing the concept of ordinary convergence for real sequences, Kostyrko, Šalát .and Wilczyński [15] introduced the concept of ideal convergence which is a generalization of statistical convergence, by using the ideal *I* of the subsets of the set of natural numbers. Some works on this field can be found in [22, 23, 24, 25] .

Throughout the article c, c_0, ℓ_∞ denote the spaces of convergent, null and bounded sequences respectively and ${}_2\ell_\infty^F$, ${}_2c^F$, ${}_2c^{I(F)}$ and ${}_2c_0^{I(F)}$ denote the classes of bounded, convergent, *I*-convergent and *I*-null fuzzy real double sequences respectively.

The notion of difference sequence space was introduced by Kizmaz [14] as follows :

$$Z(\Delta)=\{\, x=(x_k):(\Delta x_k)\in Z \,\},\ \text{for } Z = \ell_\infty,\ c \text{ and } c_0\,;$$

where $\Delta x_k = x_k - x_{k+1}$, for all $k \in N$.

The notion was further investigated by Et and Colak [5], Tripathy [18, 19], Et, Altin and Altinok [6], Esi and Isik [7], Tripathy and Esi [17], B.C. Tripathy and Baruah [21], Tripathy and Mahanta [27], Khan [13] and many others. The notion of fuzzy real-valued double difference sequence spaces of crisp set was introduced by Tripathy and Sarma [28].

Now we give the fuzzy analogues of the same as follows :

$Z(\Delta)=\{\,(X_{nk}):(\Delta X_{nk})\in Z\,\}$, for Z = ${}_2\ell_\infty{}^F$, ${}_2c^F$ and ${}_2c_0{}^F$,

where $\Delta X_{nk} = X_{n,k} - X_{n,k+1} - X_{n+1,k} + X_{n+1,k+1}$,

for all $n, k \in N$.

The aim of the present paper is to introduce and investigate some new double difference sequence spaces of fuzzy numbers and obtain some important results on them.

II. DEFINITION AND BACKGROUND

Let X be a non empty set. A non-void class $I \subseteq 2^X$ (power set of X) is said to be an ideal if I is additive

$(i.e.\ A, B\in I \Rightarrow A\cup B\in I)$ and hereditary

$(i.e.\ A\in I \text{ and } B\subseteq A \Rightarrow B\in I)$.

A non-empty family of sets $F \subseteq 2^X$ is said to be a filter on X if $\varphi \notin F$; $A, B\in F \Rightarrow A\cap B\in F$ and

$A\in F,\ A\in F,\ A\subseteq B \Rightarrow B\in F$.

For each ideal I there is a filter $F(I)$ given by $F(I)=(K\subseteq N : N\setminus K\in I\}$.

Throughout the ideals of 2^N and $2^{N\times N}$ will be denoted by I and I_2 respectively.

A fuzzy real number X is a fuzzy set on R, ie. a mapping $X: R \to L$ (= [0, 1]) associating each real number t with its grade of membership $X(t)$.

The α-level set of a fuzzy real number X, $0<\alpha\le 1$ denoted by $[X]^\alpha$, is defined as $[X]^\alpha=\{t\in R : X(t)\ge\alpha\}$.

A fuzzy real number X is called convex if $X(t)\ge X(s)\wedge X(r)=\min(X(s)\wedge X(r))$, where $s<t<r$.

If there exists $t_0\in R$ such that $X(t_0)=1$, then the fuzzy real number X is called normal. A fuzzy real number X is said to be upper semi-continuous if for each $\varepsilon>0$, $X^{-1}([0, a+\varepsilon))$, for all $a\in L$ is open in the usual topology of R. The set of all upper semi continuous, normal , convex fuzzy number is denoted by $L(R)$.

Every real number r can be expressed as a fuzzy real number $\bar{r}$ as follows **:**

$$\bar{r}(t)=\begin{cases}1 & if \quad t=r\\ 0 & otherwise\end{cases}$$

Let D be the set of all closed bounded intervals $X=[X^L, X^R]$. Then

$X\le Y$ if and only if $X^L\le Y^L$ and $X^R\le Y^R$.

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Also $d(X, Y) = \max\left(\left|X^L - Y^L\right|, \left|X^R - Y^R\right|\right)$. Then (D, d) is a complete metric space.

Let $\bar{d}: L(R) \times L(R) \to R$ be defined by

$$\bar{d}(X,Y) = \sup_{0\le\alpha\le1} d\left([X]^\alpha, [Y]^\alpha\right), \text{ for } X, Y \in L(R).$$

Then $\bar{d}$ defines a metric on $L(R)$.

Throughout a fuzzy real valued double sequence is denoted by (X_{nk}), i.e. a double infinite array of fuzzy real numbers.

A fuzzy real valued double sequence (X_{nk}) is said to be convergent in Pringsheim's sense to the fuzzy real number X, if for every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$, $k_0 = k_0(\varepsilon) \in N$ such that $\bar{d}(X_{nk}, X) < \varepsilon$ for all $n \ge n_0$, $k \ge k_0$.

A fuzzy real valued double sequence (X_{nk}) is said to be I_2-convergent to the fuzzy number X_0, if for all $\varepsilon > 0$, the $\{(n,k) \in N^2 : \bar{d}(X_{nk}, X_0) \ge \varepsilon\} \in I_2$.

We write $I_2 - \lim X_{nk} = X_0$.

A double sequence space E^F is said to be solid if $(Y_{nk}) \in E^F$, whenever $\bar{d}(Y_{nk}, \bar{0}) \le \bar{d}(X_{nk}, \bar{0})$ for all $n, k \in N$ and $(X_{nk}) \in E^F$.

A double sequence space E^F is said to be monotone if E^F contains the canonical pre-image of all its step spaces.

A double sequence space E^F is said to be symmetric if $\left(X_{\pi(n),\pi(k)}\right) \in E^F$ whenever $(X_{nk}) \in E^F$, where π is a permutation of N.

A double sequence space E^F is said to be sequence algebra if $(X_{nk} \otimes Y_{nk}) \in E^F$, whenever $(X_{nk}), (Y_{nk}) \in E^F$.

A double sequence space E^F is said to be convergence free if $(Y_{nk}) \in E^F$, whenever $(X_{nk}) \in E^F$ and $X_{nk} = \bar{0}$ implies $Y_{nk} = \bar{0}$.

Throughout (p_{nk}) is a sequence of positive real numbers such that $0 < p_{nk} \le \sup p_{nk} < \infty$.

We introduce the following sequence spaces :

$${}_2c^{I(F)}(\Delta, p) = \{X = (X_{nk}) : I_2 - \lim(\bar{d}(\Delta X_{nk}, X_0)^{p_{nk}} = 0, \text{for some } X_0 \in L(R)\}$$

$${}_2c_0^{I(F)}(\Delta, p) = \{X = (X_{nk}) : I_2 - \lim(\bar{d}(\Delta X_{nk}, \bar{0})^{p_{nk}} = 0\}$$

$${}_2\ell_\infty^F(\Delta, p) = \{X = (X_{nk}) : \sup_{n,k}(\bar{d}(\Delta X_{nk}, \bar{0}))^{p_{nk}} < \infty\}$$

Also we write

$${}_2m^{I(F)}(\Delta, p) = {}_2c^{I(F)}(\Delta, p) \cap {}_2\ell_\infty^F(\Delta, p)$$

$${}_2m_0^{I(F)}(\Delta, p) = {}_2c_0^{I(F)}(\Delta, p) \cap {}_2\ell_\infty^F(\Delta, p)$$

Lemma. If a sequence space E^F is solid , then it is monotone.

For the crisp set case, one may refer to Kamthan and Gupta [12] , p.53.

III. MAIN RESULTS

The following three results can be proved easily using simple technique.

Theorem 1. *Let $p = (p_{nk})$ be a double sequence of bounded strictly positive numbers. Then the class of sequences ${}_2m_0^{I(F)}(\Delta, p)$, ${}_2m^{I(F)}(\Delta, p)$ and ${}_2\ell_\infty^F(\Delta, p)$ are linear spaces.*

Theorem 2. *Let the double sequence $p = (p_{nk})$ be bounded. Then*

$${}_2m_0^{I(F)}(\Delta, p) \subset {}_2m^{I(F)}(\Delta, p) \subset {}_2\ell_\infty^F(\Delta, p)$$ *and the inclusions are strict.*

Theorem 3. *The class of sequences ${}_2m^{I(F)}(\Delta, p)$ and ${}_2m_0^{I(F)}(\Delta, p)$ are complete metric spaces with respect to the metric ρ defined by*

$$\rho(X,Y) = \sup_n \bar{d}(X_{n1}, Y_{n1}) + \sup_k \bar{d}(X_{1k}, Y_{1k}) + \sup_{n,k} \bar{d}(\Delta X_{nk}, \Delta Y_{nk})$$

where $X = (X_{nk})$ and $Y = (Y_{nk})$ are in ${}_2m^{I(F)}(\Delta, p)$ and ${}_2m_0^{I(F)}(\Delta, p)$ and $\Delta X_{nk} = X_{n,k} - X_{n,k+1} - X_{n+1,k} + X_{n+1,k+1}$, for all $n, k \in N$.

Theorem 4. *The class of sequence ${}_2m_0^{I(F)}(\Delta, p)$ is solid as well as monotone.*

Proof. Let $(X_{nk}) \in {}_2m_0^{I(F)}(\Delta, p)$ and (Y_{nk}) be such that $|Y_{nk}| \le |X_{nk}|$ for all $n, k \in N$.

Then the solidness of ${}_2m_0^{I(F)}(\Delta, p)$ follows from the following inclusion relation :

$$\left\{(n,k) \in N \times N : [\bar{d}(\Delta X_{nk}, \bar{0})]^{p_{nk}} \ge \varepsilon\right\} \supseteq \left\{(n,k) \in N \times N : [\bar{d}(\Delta Y_{nk}, \bar{0})]^{p_{nk}} \ge \varepsilon\right\}.$$

Also by Lemma 1, it follows that the space is monotone. ■

Theorem 5. *The class of sequences ${}_2c^{I(F)}(\Delta, p)$, ${}_2c_0^{I(F)}(\Delta, p)$, ${}_2m^{I(F)}(\Delta, p)$ and ${}_2m_0^{I(F)}(\Delta, p)$ are sequence algebras.*

Proof . We consider the space ${}_2c_0^{I(F)}(\Delta, p)$.

Let $(X_{nk}), (Y_{nk}) \in {}_2c_0^{I(F)}(\Delta, p)$ and $0 < \varepsilon < 1$. Then the result follows from the following inclusion relation:

$$\{(n,k) \in N \times N : [\bar{d}(\Delta X_{nk} \otimes \Delta Y_{nk} \bar{0})]^{p_{nk}} < \varepsilon\}$$
$$\supseteq \{(n,k) \in N \times N : [\bar{d}(\Delta X_{nk}, \bar{0})]^{p_{nk}} < \varepsilon\}$$
$$\cap \{(n,k) \in N \times N : [\bar{d}(\Delta Y_{nk}, \bar{0})]^{p_{nk}} < \varepsilon\}$$

Similarly, we can prove the result for other cases. ■

Theorem 6. *The class of sequences* ${}_2m^{I(F)}(\Delta, p)$ and ${}_2m_0^{I(F)}(\Delta, p)$ are *nowhere dense subsets of ${}_2\ell_\infty^{I(F)}(\Delta, p)$.*

Proof. By Theorem 2, the sequence spaces ${}_2m^{I(F)}(\Delta, p)$ and ${}_2m_0^{I(F)}(\Delta, p)$ are proper subspaces of ${}_2\ell_\infty^{I(F)}(\Delta, p)$. Hence the result follows from Theorem 3. ∎

IV. CONCLUSION

In this paper, we have introduced some new type of fuzzy real-valued difference double sequence space. We have studied and investigated its different algebraic and topological properties like solidness, monotonocity, sequence algebra etc. It is hoped that the properties investigated in this paper can be applied for studying the fuzzy analogue of those results..

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