On a r - GCD-Sum Function Over r-Regular Integers Modulo n^r

V. Siva Rama Prasad, P. Anantha Reddy, M.Ganeshwar Rao

Abstract: ***Introducing an r-gcd-sum function over r-regular integers modulo n^r, (studied by the authors [10] earlier), we obtain an asymptotic formula for its summatory function. The case $r=1$ of our result gives the formula established by László Tóth [8].***

Key words: ***r-regular integers modulo n^r, r-gcd of two positive integers, (n, r)-residue system, reduced (n, r)-residue system, unitary divisor of an integer, Dirichlet divisor problem, Riemann Hypothesis. 2010 Mathematics Subject Classification: Primary: 11A25, Secondary: 11N37***

I. INTRODUCTION

Let r be a fixed positive integer. A positive integer k is said to be *r-regular modulo n^r* if there is an integer x such that $k^{r+1}x \equiv k^r (\bmod n^r)$ holds. For example, the set of 2-regular integers modulo 4^2 is {1,2,3,5,6,7,9,10,11,13,14,15}.

The case $r=1$ gives the *regular integer modulo n*, a detailed study of which was made by László Tóth [7], who gave the history of such integers in the same paper. To find several equivalent conditions for an integer to be r-regular modulo n^r, the authors have used in [10] the notion of r-gcd of two positive integers given below:

For positive integers a and b, their greatest r^{th} power common divisor is denoted by $(a,b)_r$ and is called the *r-gcd of* a and b. For example, $(100,18)_2 = 1, (100,16)_2 = 4$ while $(1000,16)_3 = 8.$ Note that $(a,b)_1 = (a,b)$, the greatest common divisor of a and b. Though the concept of r-gcd is due to Fogel (see [3], p.134), a study of it was initiated by V.L.Klee [6].

In [10] the authors showed, among other things, that .

(1.1) *A positive integer k is r-* regular integers modulo n^r *if and only if* $(k, n^r)_r$ *is a unitary divisor of* n^r.

Recall that a divisor d of a positive integer m is said to be *unitary* if $\left(d, \frac{m}{d}\right) = 1$; and in this case we write $d \| m$.

Clearly $a=0$ is r-regular modulo n^r for every $n>1$ and $r \geq 1$. Also it is easy to see that if $a \equiv b (\bmod n^r)$ then a and b are r-regular modulo n^r simultaneously. Further if a and b are r-regular modulo n^r then so is ab.

The notions given in ([9], p. 42) are also needed:

A complete set of residues modulo n^r is called a *(n, r)-residue system*, while the set of all *k* in a *(n, r)*-residue system such that $(k, n^r)_r = 1$ is called a *reduced (n, r)-residue system*. Note that $C_{n,r} = \{k : 1 \leq k \leq n^r\}$ is the *minimal (n, r) - residue system* and that $R_{n,r} = \{k \in C_{n,r} : (k, n^r)_r = 1\}$ is the *minimal reduced (n, r)-residue system.*

Denoting the number of elements in $R_{n,r}$ by $\varphi_r(n^r)$, V.L.Klee [6] proved that

(1.2) $$\varphi_r(n^r) = \sum_{d | n^r} \mu_r(d) \frac{n^r}{d},$$

where μ_r is the r-analogue of the Mobius function μ defined by

(1.3) $$\mu_r(n) = \begin{cases} 1 & if \ \ n=1 \\ (-1)^t \ if & n = (p_1 p_2 .... p_t)^r \ where \\ & p_1 < p_2 < .. < p_t \ are \ primes \\ 0 & otherwise \end{cases}$$

and that

(1.4) φ_r is multiplicative.

Let $\mathrm{Reg}_r(n)$ denote the set of all r-regular integers modulo n^r in $C_{n,r}$. That is,

$$\mathrm{Reg}_r(n) = \{k \in C_{n,r} : k \ is \ r - \text{regular modulo } n^r\}$$

Manuscript received on March, 2013.

Dr. V. Siva Rama Prasad, Nalla Malla Reddy Engineering College Divyanagar, Kachivanisingaram (PO),Ghatkesar Mandal, R.R.District, Andhra Pradesh, INDIA.

P.Anantha Reddy, Government Polytechnic, Vikarabad, R.R.District, Andhra Pradesh, INDIA.

M.Ganeshwar Rao, Chaitanya Bharathi Institute of Technology, Gandipet, Hyderabad, Andhra Pradesh, INDIA.

It is easy to verify that

(1.5) $R_{n,r} \subset \mathrm{Reg}_r(n)$ for every integer $n>1$ and $r \geq 1$. Indeed, if $a \in R_{n,r}$ then $\left(a,n^r\right)_r = 1$ so that $\left(a,n^r\right)=1$ and therefore there is an integer x_0 such that $ax_0 \equiv 1(\mathrm{mod}\, n^r)$ for which $a^{r+1}x_0 \equiv a^r(\mathrm{mod}\, n^r)$ also holds.

Now if we write $\rho_r\left(\mathrm{n}^r\right) := \#\mathrm{Reg}_r(n)$, then we have

(1.6) $\varphi_r\left(n^r\right) < \rho_r\left(n^r\right) \leq n^r$ for every $n>1$ and $r \geq 1$.

Defining the r-gcd-sum function by

(1.7) $$P_r\left(n^r\right) := \sum_{k \in C_{n,r}} \left(k,n^r\right)_r ,$$

the authors [11] have established an asymptotic formula for its summatory function $\sum_{n^r \leq x} P_r\left(n^r\right)$, which gives a result due to Olivier Bordellès[1] as a particular case.

In the present paper we introduce the function

(1.8) $$\tilde{P}_r\left(n^r\right) := \sum_{k \in \mathrm{Re}\, g_r(n)} \left(k,n^r\right)_r ,$$

where the summation is over all the elements $k \in \mathrm{Reg}_r(n)$. Clearly $\tilde{P}_r\left(n^r\right) \leq P_r\left(n^r\right)$ for every $n>1$ and r ³ 1.

The aim of this paper is to obtain some arithmetic properties of this function and then find an asymptotic formula for its summatory function. Recently László Tóth [8] considered the case $r=1$ of the function given in (1.8) and results proved therein can be deduced from our results.

II. ARITHMETIC PROPERTIES

First we prove

2.1. Lemma.

(i) For every $n \geq 1$ and $r \geq 1$

$$\tilde{P}_r\left(n^r\right) = \sum_{d^r \| n^r} d^r \varphi_r\left(\frac{n^r}{d^r}\right)$$

(ii) $\tilde{P}_r\left(n^r\right)$ is multiplicative in n

and

(iii) $\tilde{P}_r\left(p^{\alpha r}\right) = 2p^{\alpha r} - p^{(\alpha-1)r}$,

for every prime p and integer $\alpha \geq 1$.

Proof: (i) In view of (1.1), for each $k \in \mathrm{Reg}_r(n)$ there is a unique $d^r \| n^r$ such that $d^r = \left(k,n^r\right)_r$.

Also for each $d^r \| n^r$ the number of such k's is equal to number of j's with $1 \leq j \leq \dfrac{n^r}{d^r}$ and $\left(j,\dfrac{n^r}{d^r}\right)_r = 1$. This number is equal to $\varphi_r\left(\dfrac{n^r}{d^r}\right)$. Hence $\tilde{P}_r\left(n^r\right) = \sum_{d^r \| n^r} d^r \varphi_r\left(\dfrac{n^r}{d^r}\right)$, proving part (i)

(ii) If $\chi_r(m)=1$ or 0 according as m is the r^{th} power of an integer or not; and if $E(m)=m$ for every integer $m \geq 1$ then the identity can be written as

(2.2) $$\tilde{P}_r\left(n^r\right) = \sum_{D \| n^r} \chi_r(D) E(D) \varphi_r\left(\frac{n^r}{D}\right)$$

$$= \sum_{D \| n^r} f(D) \varphi_r\left(\frac{n^r}{D}\right)$$

where $f(m) = \chi_r(m)E(m)$ for every $m \geq 1$. That is, $\tilde{P}_r\left(n^r\right) = \left(f \circ \varphi_r\right)\left(n^r\right)$, where $\circ$ is the unitary convolution of arithmetic functions studied by Eckford Cohen [2], (In fact, for arithmetic functions g and h their unitary convolution $g \circ h$ is defined by $\left(g \circ h\right)(n) = \sum_{d \| n} g(d) h\left(\dfrac{n}{d}\right)$). It has been proved in [2] that whenever g and h are multiplicative so is $g \circ h$. Hence $\tilde{P}_r = f \circ \varphi_r$ is also multiplicative.

(iii) Further if p is a prime and α is an integer ≥ 1 then

$$\tilde{P}_r\left(p^{\alpha r}\right) = \sum_{d^r \| p^{\alpha r}} d^r \varphi_r\left(\frac{p^{\alpha r}}{d^r}\right) = \varphi_r\left(p^{\alpha r}\right) + p^{\alpha r}\varphi_r(1)$$

$$= \left(p^{\alpha r} - p^{(\alpha-1)r}\right) + p^{\alpha r}$$

$$= 2p^{\alpha r} - p^{(\alpha-1)r}.$$

2.3. Remark. If $n>1$ has canonical representation $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$ then Lemma 2.1 shows that

$$\tilde{P}_r\left(p^{\alpha r}\right) = \left(2p_1^{\alpha_1 r} - p_1^{(\alpha_1-1)r}\right)\left(2p_2^{\alpha_2 r} - p_2^{(\alpha_2-1)r}\right) \ldots$$

$$\left(2p_t^{\alpha_t r} - p_t^{(\alpha_t-1)r}\right)$$

$$= n^r\left(2-\frac{1}{p_1^r}\right)\left(2-\frac{1}{p_2^r}\right)\cdots\left(2-\frac{1}{p_t^r}\right)$$

$$= n^r\prod_{p\mid n}\left(2-\frac{1}{p^r}\right)$$

Using Lemma 2.1, we find another representation of $\widetilde{P}_r\left(n^r\right)$ given below:

2.4. Lemma. For any $n \geq 1$ and integer $r \geq 1$,

$$\widetilde{P}_r\left(n^r\right)=\sum_{d\delta=n^r}\mu_r(d)\delta\tau^*(\delta;d),$$

where $\tau^*(m;k)$ denotes the number of unitary divisors of m which are relatively prime to k (In symbols, $\tau^*(m;k)=\sum_{\substack{d\|m\\(d,k)=1}}1$).

Proof: - By part (i) of Lemma 2.1 and (1.2),

$$\widetilde{P}_r\left(n^r\right)=\sum_{\substack{t^r u^r=n^r\\(t^r,u^r)=1}} t^r\varphi_r\left(u^r\right)$$

$$=\sum_{\substack{t^r u^r=n^r\\(t^r,u^r)=1}} t^r\left(\sum_{dv=u^r}\mu_r(d)v\right)$$

$$=\sum_{\substack{Du^r=n^r\\(D,u^r)=1}} D\left(\sum_{dv=u^r}\mu_r(d)v\right)$$

$$=\sum_{\substack{Ddv=n^r\\(D,dv)=1}} D\mu_r(d)v$$

$$=\sum_{d\delta=n^r}\mu_r(d)\delta\left(\sum_{\substack{Dv=\delta\\(D,d)=1\\(D,v)=1}}1\right)$$

$$=\sum_{d\delta=n^r}\mu_r(d)\delta\left(\sum_{\substack{D\|\delta\\(D,d)=1}}1\right)$$

$$=\sum_{d\delta=n^r}\mu_r(d)\delta\tau^*(\delta;d).$$

3. Asymptotic formula for $\widetilde{S}_r(x):=\sum_{n^r\leq x}\widetilde{P}_r\left(n^r\right)$.

In this section we deduce an asymptotic formula for the summatory function

$\widetilde{S}_r(x):=\sum_{n^r\leq x}\widetilde{P}_r\left(n^r\right)$ in two cases- one without assuming the Riemann Hypothesis and the other with its assumption.

We use the following notation and also some preliminary lemmas given in [12] and [8]:

The Dedikind function ψ is defined (see [9], p.41) by

(3.1) $$\psi(n)=n\prod_{p\mid n}\left(1+\frac{1}{p}\right),$$

where the product is over the primes dividing n.

Set

(3.2) $\alpha(k)=\sum_{p\mid k}\frac{\log p}{p-1}$ for any integer $k > 1$ and $\alpha(1)=0$,

(3.3) $\beta(k)=\sum_{p\mid k}\frac{\log p}{p^2-1}$ for any integer $k > 1$ and $\beta(1)=0$,

(3.4) $$\delta(x)=\exp\left\{-A(\log x)^{3/5}.(\log\log x)^{-1/5}\right\}$$

and

(3.5) $$\eta(x)=\exp\left\{B(\log x).(\log\log x)^{-1}\right\},$$

where A and B are positive constants.

Note that $\alpha(k)\leq\sum_{p\mid k}\log p=\log k$ while $\beta(k)\leq\sum_{p\mid k}\frac{\log p}{p^2}<\infty$

so that

(3.6) $$\alpha(k)=O(\log k) \text{ and } \beta(k)=O(1).$$

(3.7) It is well-known that the *Dirichlet divisor problem* seeks the least positive real number θ for which

$\sum_{n\leq x}\tau(n)=x(\log x+2\gamma-1)+O_\varepsilon\left(x^{\theta+\varepsilon}\right)$ holds for each $x\geq 1$ and any $\varepsilon>0$.

Hardy [4] proved that $\theta\geq\frac{1}{4}$, while the best upper bound for θ given Huxley [5] is $\theta\leq\frac{131}{416}$.

3.8 Lemma ([12], Theorem 4.3). For any integer $k\geq 1$ and every $\varepsilon>0$,

$$\sum_{m\le x}\tau^*(m;k)=\frac{kx}{\zeta(2)\psi(k)}\Big(\log x+\alpha(k)-2\beta(k)+ 2\gamma-1-\frac{2\zeta'(2)}{\zeta(2)}\Big)+O\Big(\sigma'_{-1+\varepsilon}(k)\sigma'_{-\theta}(k)x^{1/2}\delta(x)\Big)$$

where $\psi(k)$, $\alpha(k)$, $\beta(k)$ and $\delta(x)$ are given in (3.1) to (3.4); and $\sigma'_s(k)$ is the sum of the s-th power of square free divisor of k.

3.9 Lemma ([12], Theorem 5.2). If the Riemann Hypothesis is true then the error term in the asymptotic formula in Lemma 3.8 is $O\Big(\sigma'_{-1+\varepsilon}(k)\sigma'_{-\theta}(k)x^{(2-\theta)/(5-4\theta)}\eta(x)\Big)$, where $\eta(x)$ is given in (3.5).

Using Lemma 3.8 and Lemma 3.9; and the Abel's identity László Tóth [8] derived the following:

3.10 Lemma ([12], Lemma2, p.6). For every $\varepsilon>0$,

$$\sum_{m\le x}m\tau^*(m;k)=\frac{kx^2}{2\zeta(2)\psi(k)}\Big(\log x+\alpha(k)-2\beta(k) +2\gamma-\frac{1}{2}-2\frac{\zeta'(2)}{\zeta(2)}\Big)+O\Big(\sigma'_{-1+\varepsilon}(k)\sigma'_{-\theta}(k)x^{3/2}\delta(x)\Big)$$

and if the Riemann Hypothesis is true then the error term in this formula can be replaced by $O\Big(\sigma'_{-1+\varepsilon}(k)\sigma'_{-\theta}(k).x^{(7-5\theta)/(5-4\theta)}\eta(x)\Big)$.

Our main theorem is

3.11. Theorem.

(i) For $x\ge 1$,

$$\tilde{S}_r(x)=\frac{x^2}{2z(2)}(M\log x+N)+O\Big(x^{3/2}d(x)\Big),$$

where

$$\text{(3.12)}\qquad M=\sum_{n=1}^{\infty}\frac{\mu_r(n)}{n\psi(n)}=\prod_p\Big\{1-\frac{1}{p^{2r-1}(p+1)}\Big\}$$

and

$$\text{(3.13)}\qquad N=M\Big(2\gamma-\frac{1}{2}-2\frac{\zeta'(2)}{\zeta(2)}\Big) -\sum_{n=1}^{\infty}\frac{\mu_r(n)(\log n-\alpha(n)+2\beta(n))}{n\psi(n)}.$$

(ii) If the Riemann Hypothesis is true the error term in the above formula is replaced by

$$O\Big(x^{(7-5\theta)/(5-4\theta)}\eta(x)\Big).$$

Proof: - By Lemma 2.4 and Lemma 3.10,

$$\text{(3.14)}\qquad \tilde{S}_r(x)=\sum_{d\delta\le x}\mu_r(d)\delta\tau^*(\delta;d)$$

$$=\sum_{d\le x}\mu_r(d)\Big\{\sum_{\delta\le\frac{x}{d}}\delta\tau^*(\delta;d)\Big\}$$

$$=\sum_{d\le x}\mu_r(d)\Big\{\frac{d\left(\frac{x}{d}\right)^2}{2\zeta(2)\psi(d)}\Big(\log\Big(\frac{x}{d}\Big)+\alpha(d)-2\beta(d)+ 2\gamma-\frac{1}{2}-2\frac{\zeta'(2)}{\zeta(2)}\Big)+O\Big(\sigma'_{-1+\varepsilon}(d)\sigma'_{-\theta}(d)\Big(\frac{x}{d}\Big)^{3/2}\delta\Big(\frac{x}{d}\Big)\Big)\Big\}$$

$$=\frac{x^2}{2\zeta(2)}\sum_{d\le x}\frac{\mu_r(d)}{d\psi(d)}\Big(\log x-\log d+\alpha(d)-2\beta(d)+ 2\gamma-\frac{1}{2}-2\frac{\zeta'(2)}{\zeta(2)}\Big)$$

$$+O\Big(\sum_{d\le x}|\mu_r(d)|\sigma'_{-1+\varepsilon}(d)\sigma'_{-\theta}(d)\Big(\frac{x}{d}\Big)^{3/2}\delta\Big(\frac{x}{d}\Big)\Big)$$

$$=\frac{x^2}{2\zeta(2)}\Big(\log x+2\gamma-\frac{1}{2}-2\frac{\zeta'(2)}{\zeta(2)}\Big)\sum_{d\le x}\frac{\mu_r(d)}{d\psi(d)}$$

$$+\frac{x^2}{2\zeta(2)}\sum_{d\le x}\frac{(\alpha(d)-2\beta(d)-\log d)\mu_r(d)}{d\psi(d)}$$

$$+O\Big(\sum_{d\le x}\sigma'_{-1+\varepsilon}(d)\sigma'_{-\theta}(d)\Big(\frac{x}{d}\Big)^{3/2}\delta\Big(\frac{x}{d}\Big)\Big)$$

$$=\frac{x^2}{2\zeta(2)}\Big(\log x+2\gamma-\frac{1}{2}-2\frac{\zeta'(2)}{\zeta(2)}\Big)\cdot\Big(\sum_{n=1}^{\infty}\frac{\mu_r(n)}{n\psi(n)}-\sum_{n>x}\frac{\mu_r(n)}{n\psi(n)}\Big)$$

$$+\frac{x^2}{2\zeta(2)}\Big(\sum_{n=1}^{\infty}\frac{(\alpha(n)-2\beta(n)-\log n)\mu_r(n)}{n\psi(n)} -\sum_{n>x}\frac{(\alpha(n)-2\beta(n)-\log n)\mu_r(n)}{n\psi(n)}\Big)$$

$$+O\Big(\sum_{n\le x}\sigma'_{-1+\varepsilon}(n)\sigma'_{-\theta}(n)\Big(\frac{x}{n}\Big)^{3/2}\delta\Big(\frac{x}{n}\Big)\Big)$$

$$=\frac{x^2}{2\zeta(2)}\Big\{M\Big(\log x+2\gamma-\frac{1}{2}-2\frac{\zeta'(2)}{\zeta(2)}\Big)$$

$$+\sum_{n=1}^{\infty}\frac{\mu_r(n)[\alpha(n)-2\beta(n)-\log n]}{n\psi(n)}\Bigg\}$$

$$+O\left(x^2\log x.\sum_{n>x}\frac{|\mu_r(n)|}{n\psi(n)}\right)$$

$$+O\left(x^2.\sum_{n>x}\frac{|\mu_r(n)||\alpha(n)-2\beta(n)-\log n|}{n\psi(n)}\right)$$

$$+O\left(\sum_{n\le x}\sigma'_{-1+\varepsilon}(n)\sigma'_{-\theta}(n)\left(\frac{x}{n}\right)^{3/2}\delta\left(\frac{x}{n}\right)\right).$$

In view of (3.1),

(3.15) $$\sum_{n>x}\frac{|\mu_r(n)|}{n\psi(n)}\le\sum_{n>x}\frac{1}{n^2}=O\left(\frac{1}{x}\right).$$

Also by (3.1) and (3.6) we get $|\mu_r(n)||\alpha(n)-2\beta(n)-\log n|=O(\log n)$ so that

(3.16)

$$\sum_{n>x}\frac{|\mu_r(n)||\alpha(n)-2\beta(n)-\log n|}{n\psi(n)}\le\sum_{n>x}\frac{\log n}{n^2}=O\left(\frac{\log x}{x}\right)$$

Finally, since $x^{\varepsilon}\delta(x)$ is increasing, we have

$$\left(\frac{x}{n}\right)^{\frac{3}{2}}\delta\left(\frac{x}{n}\right)=\left(\frac{x}{n}\right)^{\frac{3}{2}-\varepsilon}\left(\frac{x}{n}\right)^{\varepsilon}\delta\left(\frac{x}{n}\right)\le\left(\frac{x}{n}\right)^{\frac{3}{2}-\varepsilon}x^{\varepsilon}\delta(x)$$

$$=\frac{x^{\frac{3}{2}}}{n^{\frac{3}{2}-\varepsilon}}\delta(x)$$

and since $\sigma'_{-1+\varepsilon}(n)\le\tau(n),\ (\text{for }\varepsilon<1),$

$\sigma'_{-\theta}(n)\le\tau(n)$ we get that

(3.17) $$O\left(\sum_{n\le x}\sigma'_{-1+\varepsilon}(n)\sigma'_{-\theta}(n)\left(\frac{x}{n}\right)^{3/2}\delta\left(\frac{x}{n}\right)\right)$$

$$=O\left(\sum_{n\le x}\frac{\tau^2(n).x^{\frac{3}{2}}\delta(x)}{n^{\frac{3}{2}-\varepsilon}}\right)$$

$$=O\left(x^{\frac{3}{2}}\delta(x)\sum_{n\le x}\frac{\tau^2(n)}{n^{\frac{3}{2}-\varepsilon}}\right)$$

$$=O\left(x^{\frac{3}{2}}\delta(x)\right)$$

Now using (3.15), (3.16), (3.17), (3.12) and (3.13) in (3.14) we get

(3.18) $$\tilde{S}_r(n^r)=\frac{x^2}{2\zeta(2)}\{M\log x+N\}+O\left(x^2\log x.\frac{1}{x}\right)$$

$$+O\left(x^2.\frac{\log x}{x}\right)+O\left(x^{3/2}\delta(x)\right)$$

$$=\frac{x^2}{2\zeta(2)}\{M\log x+N\}+O\left(x^{3/2}\delta(x)\right),$$

proving part (i) of the theorem.

If the Riemann Hypothesis is true, then (by the Lemma3.10) the only change in the above is in the error term given in 3.17 and it is replaced by

$$O\left(\sum_{n\le x}\sigma'_{-1+\varepsilon}(n)\sigma'_{-\theta}(n)\left(\frac{x}{n}\right)^{(7-5\theta)/(5-4\theta)}\eta\left(\frac{x}{n}\right)\right).$$ Here since $\eta(x)$ is increasing, we get

$$\sum_{n\le x}\sigma'_{-1+\varepsilon}(n)\sigma'_{-\theta}(n)\left(\frac{x}{n}\right)^{(7-5\theta)/(5-4\theta)}\eta(x)$$

$$\le x^{\frac{7-5\theta}{5-4\theta}}\eta(x)\sum_{n\le x}\frac{\tau^2(n)}{n^{(7-5\theta)/(5-4\theta)}}=O\left(x^{\frac{7-5\theta}{5-4\theta}}\eta(x)\right),$$

since $\frac{7-5\theta}{5-4\theta}>1$ implies the partial sum on the right is bounded.

Now the dominant among error terms on the right of (3.18) is $O\left(x^{\frac{7-5\theta}{5-4\theta}}\eta(x)\right)$ and hence the theorem follows in this case also.

3.19 Remark. The case $r=1$ of Theorem 3.11 gives an asymptotic formula for $\sum_{n\le x}\tilde{P}(n)$, where $\tilde{P}(n)=\sum_{k\in\operatorname{Re}g_1(n)}(k,n)$, obtained by László Tóth [8], since $\mu_1(n)=\mu(n)$ for each $n\ge1$.

REFERENCES

1. O. Bordellès, A note on the average order of the gcd-sum function, J. Integer Sequences, 10 (2007), Art. 07.3.3
2. E. cohen, Arithmetical functions associated with the unitary divisors of an integer, Math. Zeit. , 74(1960), 66-80
3. L.E. Dickson, History of Theory of numbers, Volume I, Carnegie Institution of Washington, 1919; reprinted Chelsea Publishing Company, New York, 1952
4. G.H. Hardy, The average order of the arithmetical functions $P(x)$ and $\Delta(x)$, Proc., London Math. Soc., 15(2), (1916), 192-213
5. M.N. Huxley, Exponential sums and Lattice points III, Proc. London Math. Soc., 87 (2003), 591-609
6. V.L. Klee, A generalization of Euler's function, Amer. Math. Monthly, 55 (1948), 358-359
7. László Tóth, Regular integers (mod n), Annales Univ. Sci. Budapest. Sect. Comp., 29 (2008), 263-275
8. László Tóth, A gcd-sum function over regular integers modulo n, J. Integer Sequences, Vol.12 (2009), Article 09.2.5
9. Paul J. McCarthy, Introduction to Arithmetical functions, Springer-Verlag, New York, 1986
10. V. Siva Rama Prasad, P. Anantha Reddy, and M. Ganeshar Rao, r-Regular Integers Modulo n^r, J. Andra Pradesh Society of Math. Sci. (Accepted)
11. V. Siva Rama Prasad, P. Anantha Reddy, and M. Ganeshar Rao, On the r-gcd-sum function (communicated).
12. D. Suryanarayana and V. Siva Rama Prasad, The number of k-free and k-ary divisors of m which are prime to n, J. Reine Angew. Math. 264 (1973), 56-75