Fuzzy Inventory Model for Deteriorating Items with Shortages under Fully Backlogged Condition

D. Dutta, Pavan Kumar

Abstract— ***In this paper, a fuzzy inventory model for deteriorating items with shortages under fully backlogged condition is formulated and solved. Deterioration rate and demand are assumed to be constant. Shortages are allowed and assumed to be fully backlogged. Fuzziness is introduced by allowing the cost components (holding cost, shortage cost, etc.), demand rate and the deterioration. In fuzzy environment, all related inventory parameters are assumed to be trapezoidal fuzzy numbers. The purpose of this paper is to minimize the total cost function in fuzzy environment. A numerical example is given in order to show the applicability of the proposed model. The convexity of the cost function is shown graphically. Sensitivity analysis is also carried out to detect the most sensitive parameters of the system. From sensitivity analysis, we show that the total cost function is extremely influenced by the holding cost, demand rate and the shortage cost.***

Keywords—Inventory model, Trapezoidal fuzzy number, Fuzzy demand, Fuzzy deterioration.

I. INTRODUCTION

In 1915, the first inventory model was developed by F. Harris [1]. Later in 1965, first time the concept of fuzzy sets was introduced by Lotfi A Zadeh [2]. Fuzzy set theory is an extension of classical set theory where elements have degrees of membership. The theory of fuzzy sets attracted the attention of many researches. In 1970, L. A. Zadeh and R. E. Bellman proposed a mathematical model on decision making in a fuzzy environment [3]. In 1976, a fuzzy model on decision making in the presence of fuzzy variables was proposed by R. Jain [4]. Later in 1978, D. Dubois and H. Prade [5] defined some operations on fuzzy numbers. In 1983, H. J. Zimmerman [7] made an attempt to use the fuzzy sets in operational research.

Some researchers started to apply fuzzy set theory in inventory management problems. In 1982, J. Kacpryzk and P. Staniewski [6] proposed a model on long-term inventory policy-making through fuzzy-decision making models. In 1983, G. Urgeletti Tinarelli [8] proposed the inventory control models and problems. In 1987, K. S. Park [9] proposed a model on fuzzy set theoretical interpretation of economic order quantity inventory problem. In 1996, M. Vujosevic, D. Petrovic and R. Petrovic [10] developed an EOQ formula when inventory cost is a fuzzy number. In 1999, J. S. Yao and H. M. Lee [11] proposed a fuzzy inventory with or without backorder for fuzzy order quantity with trapezoidal fuzzy number. In 1999, J. S. Yao and H. M. Lee [12] developed an economic order quantity model in fuzzy sense for inventory without backorder model. In 2002, C. K. Kao and W. K. Hsu [13] proposed a single-period inventory model with fuzzy demand. In 2002, C. H. Hsieh [14] proposed an approach for the optimization of fuzzy production inventory models. In 2003, J. S. Yao and J. Chiang [15] introduced an inventory without back order with fuzzy total cost and fuzzy storing cost defuzzified by centroid and signed distance. They compared the optimal results obtained by both the defuzzification methods.

The effect of deterioration is very important in many inventory systems. Deterioration is defined as decay or damage such that the item cannot be used for its original purpose. Most of the physical goods undergo decay or deterioration over time. Commodities such as fruits, vegetables, foodstuffs, etc., suffer from depletion by direct spoilage while kept in store. Highly volatile liquids such as gasoline, alcohol, turpentine, etc., undergo physical depletion over time through the process of evaporation. In the development economic production lot size models, usually researchers consider the deterioration rate, demand rate, unit cost, etc., as fixed, but all of them probably will have some little fluctuations for each cycle in real life situation. So in practical situations, if these quantities are treated as fuzzy variables then it will be more realistic. In 2003, Sujit De Kumar, P. K. Kundu and A. Goswami [16] presented an economic production quantity inventory model involving fuzzy demand rate and fuzzy deterioration rate.

In 2007, J. K. Syed and L. A. Aziz [17] applied signed distance method to Fuzzy inventory model without shortages. In 2011, P. K. De and A. Rawat [18] proposed a fuzzy inventory model without shortages using triangular fuzzy number. In 2012, C. K. Jaggi, S. Pareek, A. Sharma and Nidhi [19] presented a fuzzy inventory model for deteriorating items with time-varying demand and shortages. In 2012, Sumana Saha and Tripti Chakrabarti [20] proposed a fuzzy EOQ model for time dependent deteriorating items and time dependent demand with shortages. Very recently, D. Dutta and Pavan Kumar published several papers in the area of fuzzy inventory with or with shortages. In 2012, presented a fuzzy inventory model without shortage using trapezoidal fuzzy number with sensitivity analysis [21]. In 2013, the same authors D. Dutta and Pavan Kumar [22] proposed an optimal policy for an inventory model without shortages considering fuzziness in demand, holding cost and ordering cost.

In this paper, we first consider a crisp inventory model with constant deteriorating items with constant demand where shortages are allowed with fully backlogged condition.

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Thereafter we develop the corresponding fuzzy inventory model for fuzzy deteriorating items with fuzzy demand rate under fully backlogging. The average total inventory cost in fuzzy sense is derived. All inventory parameters including deterioration rate are fuzzified as the trapezoidal fuzzy numbers. The fuzzy model is defuzzified by using the signed distance method (D. Dutta and Pavan Kumar [21]). The solution for minimizing the fuzzy cost function has been derived.

This paper is organized as follows: In section II, we give some necessary definitions. In section III, we describe in brief the notations and assumptions used in the developed model. In section IV we present the mathematical model and in section V we present the corresponding Fuzzy model & solution procedure. In section VI, a numerical example is given to illustrate the model. In section VII, Sensitivity analysis has been made for different changes in the parameter values and in section VIII we do some observations. Finally, conclusions are given in section IX.

II. DEFINITIONS AND PRELIMINARIES [2, 3, 7]

A fuzzy set $\tilde{A}$ on the given universal set X is a set of ordered pairs

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$$

where $\mu_{\tilde{A}} : X \to [0, 1]$, is called membership function. The α – cut of $\tilde{A}$, is defined by $A_\alpha = \{x : \mu_{\tilde{A}}(x) = \alpha, \alpha \geq 0\}$

If R is the real line, then a fuzzy number is a fuzzy set $\tilde{A}$ with membership function $\mu_{\tilde{A}} : X \to [0, 1]$, with the following properties:

(i) $\tilde{A}$ is normal, i.e., there exists $x \in R$ such that $\mu_{\tilde{A}}(x) = 1$.

(ii) $\tilde{A}$ is piece-wise continuous.

(iii) supp ($\tilde{A}$) = cl{ $x \in R : \mu_{\tilde{A}}(x) > 0$}, where cl represents the closure of a set.

(iv) $\tilde{A}$ is a convex fuzzy set.

Definition 2.1: A trapezoidal fuzzy number $\tilde{A}$ = (a, b, c, d) is represented with membership function $\mu_{\tilde{A}}$ as:

$$\mu_{\tilde{A}}(x) = \begin{cases} L(x) = \frac{x-a}{b-a}, & \text{when } a \leq x \leq b \\ 1, & \text{when } b \leq x \leq c \\ R(x) = \frac{d-x}{d-c}, & \text{when } c \leq x \leq d \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.2: A fuzzy set is called in LR-Form, if there exist reference functions L (for left), R (for right), and scalars m > 0 and n > 0 with membership function

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{\sigma-x}{m}\right), & \text{for } x \leq \sigma \\ 1, & \text{for } \sigma \leq x \leq \gamma \\ R\left(\frac{x-\gamma}{n}\right), & \text{for } x \geq \gamma. \end{cases}$$

Where σ is a real number called the mean value of $\tilde{A}$, m and n are called the left and right spreads, respectively. The functions L and R map $R^+ \to [0, 1]$, and are decreasing. A LR-Type fuzzy number can be represented as $\tilde{A} = (\sigma, \gamma, m, n)_{LR}$.

Definition 2.3: Suppose $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ are two trapezoidal fuzzy numbers, then arithmetical operations are defined as:

$\tilde{A} \oplus \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$

$\tilde{A} \otimes \tilde{B} = (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4)$

$\tilde{A} \ominus \tilde{B} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$

$\tilde{A} \oslash \tilde{B} = \left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}\right)$

$$\alpha \otimes \tilde{A} = \begin{cases} (\alpha a_1, \alpha a_2, \alpha a_3, \alpha a_4), & \alpha \geq 0 \\ (\alpha a_4, \alpha a_3, \alpha a_2, \alpha a_1), & \alpha < 0 \end{cases}$$

Definition 2.4: Let $\tilde{A}$ be a fuzzy set defined on R. Then the signed distance of $\tilde{A}$ is defined as:

$$d(\tilde{A}, 0) = \frac{1}{2}\int_0^1 [A_L(\alpha) + A_R(\alpha)]\, d\alpha$$

where $A_\alpha = [A_L(\alpha), A_R(\alpha)] = [a + (b - a)\alpha, d - (d - c)\alpha]$, $\alpha \in [0, 1]$, is α – cut of fuzzy set $\tilde{A}$, which is a close interval.

Remark: The signed distance d(*a*, 0) = *a*, for all a, 0 ∈ R. The meaning of above Definition is as the follows, if 0 < a then the distance between a and 0 is d(*a*, 0) = *a*. If *a* < 0 then the distance between a and 0 is – d(*a*, 0) = – *a*. Therefore, we call d(*a*, 0) = *a* is the signed distance between *a* and 0.

III. NOTATIONS AND ASSUMPTIONS

The proposed inventory model is developed under the following notations and assumptions:

Notations

C_o : Ordering cost per order.

C_h : Holding cost per unit per unit time.

C_s : Shortage cost per unit time.

C_p : purchasing cost per unit per unit time.

$R(t)$: Demand rate at any time t per unit time.

$\theta(t)$: Deterioration rate function = θ, 0 < θ <<1.

T : Length of each ordering cycle.

Q : Order quantity per unit.

C_{TS} : Total shortage cost per unit time.

$\tilde{C}_{TS}$: Fuzzy total shortage cost per unit time.

$(\tilde{C}_{TS})_{ds}$: Defuzzified value of fuzzy number $\tilde{C}_{TS}$ by using Signed distance method.

$C(t_1, T)$: Total inventory cost per unit time.

$\tilde{C}(t_1, T)$: Fuzzy total inventory cost per unit time.

$C_{ds}(t_1, T)$: Defuzzified value of fuzzy number $\tilde{C}(t_1, T)$ by using Signed distance method.

$\tilde{U}$: Fuzzified value (fuzzy number) of *U*, where *U* is any crisp number.

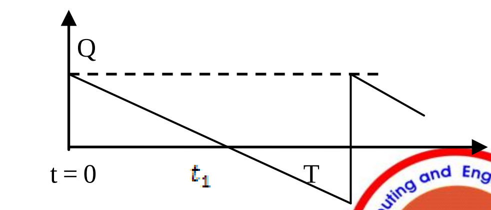
Assumptions

1. The inventory system involves only one item.
2. Demand rate R(t) = r, constant.
3. The replenishment rate is infinite, and lead time is zero.
4. Shortages are allowed and fully backlogged. Thereby, the lost sale cost per cycle is zero.
5. The deterioration rate function, θ(t), denotes the on-hand inventory deteriorates per unit time and there is no replacement or repair of deteriorated units during the period T.
6. There is no repair or replacement of the deteriorated items during the production cycle.
7. The goal of this model is to search for the optimal values of the parameters: $C(t_1, T)$, t_1 and T.

IV. MATHEMATICAL MODEL

The status of inventory is shown in Figure1.

Inventory I(t)



Time

Figure 1: Graphical Representation of Proposed Inventory System.

Let Q be the total amount of inventory purchased or produced at the beginning of each period and after fulfilling backorders. Due to reasons of market demand and deterioration of the items, the inventory level gradually diminishes during the period $[0, t_1]$, and ultimately falls to zero at $t = t_1$. Now, during time period, shortages occur which are completely backlogged.

Let I(t) be the on-hand inventory level at any time t, which is governed by the following two differential equations:

$$\frac{dI(t)}{dt} + \theta I(t) = -r, \quad \text{for } 0 \le t \le t_1 \qquad (1)$$

$$\frac{dI(t)}{dt} = -r, \quad \text{for } t_1 \le t \le T \qquad (2)$$

$$\text{with } I(0) = Q, \text{ and } I(t_1) = 0. \qquad (3)$$

Now, solve (1) and (2) using given conditions in (3). The final solution is given by

$$I(t) = \frac{r}{\theta}[e^{(t_1 - t)\theta} - 1], \quad \text{for } 0 \le t \le t_1 \qquad (4)$$

$$\text{and } \quad I(t) = r[t_1 - t], \quad \text{for } t_1 \le t \le T \qquad (5)$$

Using exponential series expansion in (4), and since $0 < \theta << 1$, so ignoring higher power term of θ, we obtain

$$I(t) = \frac{r}{\theta}[1 + (t_1 - t)\theta + (t_1 - t)^2\theta^2 - 1], \quad \text{for } 0 \le t \le t_1$$

$$= r(t_1 - t + \theta t_1^2 + \theta t^2 - 2\theta t_1 t), \text{ for } 0 \le t \le t_1 \qquad (6)$$

Using the initial condition: I(0) = Q, in (6), we obtain

$$Q = r(t_1 + \theta t_1^2), \qquad (7)$$

Total average no. of holding units (I_H) during the period [0, T] is given by

$$I_H = \int_0^{t_1} I(t)dt = r\left(\frac{1}{2}t_1^2 + \frac{1}{3}\theta t_1^3\right) \qquad (8)$$

Total no. of deteriorated units (I_D) during the period [0, T] is given by

$$I_D = Q - \text{Total Demand}$$

$$= Q - \int_0^{t_1} r\,dt = r\theta t_1^2, \qquad (9)$$

Total average no. of shortage units (I_S) during the period [0, T] is given by

$$I_S = -\int_{t_1}^{T} I(t)dt$$

$$= \frac{-r}{2}(2Tt_1 - T^2 - t_1^2) = \frac{r}{2}(T - t_1)^2 \qquad (10)$$

Total shortage cost per unit time

$$C_{TS} = \frac{1}{T}[C_s I_S]$$

$$= \frac{(T-t_1)^2}{2T} rC_s \qquad (11)$$

Total cost of the system per unit time is given by

$$C(t_1, T) = \frac{1}{T}[C_o + C_h I_H + C_p I_D + C_s I_S]$$

$$= \frac{1}{T}\left[C_o + rC_h\left(\frac{1}{2}t_1^2 + \frac{1}{3}\theta t_1^3\right) + rC_p\theta t_1^2 + \frac{r}{2}C_s(T - t_1)^2\right] \qquad (12)$$

To minimize total cost function per unit time $C(t_1, T)$, the optimal value of t_1 and T can be obtained by solving the following equations:

$$\frac{\partial C(t_1,T)}{\partial t_1} = 0, \text{ and } \frac{\partial C(t_1,T)}{\partial T} = 0$$

Now, $\frac{\partial C(t_1,T)}{\partial t_1} = \frac{1}{T}[rC_h(t_1 + \theta t_1^2) + 2rC_p\theta t_1 - rC_s(T - t_1)] = 0$

$$\Rightarrow rC_h(t_1 + \theta t_1^2) + 2rC_p\theta t_1 - rC_s(T - t_1) = 0$$

$$\Rightarrow C_h(t_1 + \theta t_1^2) + 2C_p\theta t_1 - C_s(T - t_1) = 0 \qquad (13)$$

and $\frac{\partial C(t_1,T)}{\partial T} = -\frac{1}{T^2}\left[C_o + rC_h\left(\frac{1}{2}t_1^2 + \frac{1}{3}\theta t_1^3\right) + rC_p\theta t_1^2 + \frac{r}{2}C_s(T - t_1)^2\right] + \frac{1}{T}rC_s(T - t_1) = 0$

$$\Rightarrow C_o + rC_h\left(\frac{1}{2}t_1^2 + \frac{1}{3}\theta t_1^3\right) + rC_p\theta t_1^2 + \frac{r}{2}C_s(T - t_1)^2 - T\,rC_s(T - t_1) = 0$$

$$\Rightarrow C_o + rC_h\left(\frac{1}{2}t_1^2 + \frac{1}{3}\theta t_1^3\right) + rC_p\theta t_1^2 + rC_s(T - t_1)\left(\frac{T - t_1}{2} - T\right) = 0$$

$$\Rightarrow C_o + rC_h\left(\frac{1}{2}t_1^2 + \frac{1}{3}\theta t_1^3\right) + rC_p\theta t_1^2 - \frac{1}{2}rC_s(T^2 - t_1^2) = 0 \qquad (14)$$

We solve the non linear equations (13) and (14) by using computer software MATHEMATICA 8.0. With the help of graph, we can easily prove the convexity of total cost function $C(t_1, T)$, (see Figure 2 in section VI).

V. FUZZY MODEL AND SOLUTION PROCEDURE

We consider the model in fuzzy environment. Due to fuzziness, it is not easy to define all the parameters precisely. Let $\tilde{C}_h = (C_{h1}, C_{h2}, C_{h3}, C_{h4})$, $\tilde{C}_p = (C_{p1}, C_{p2}, C_{p3}, C_{p4})$, $\tilde{C}_s = (C_{s1}, C_{s2}, C_{s3}, C_{s4})$, $\tilde{r} = (r_1, r_2, r_3, r_4)$, $\tilde{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)$, be trapezoidal fuzzy numbers in LR form. Then, the total cost of the system per unit time in fuzzy sense is given by

$$\tilde{C}(t_1, T) = \frac{1}{T}\left[C_o + \tilde{r}\tilde{C}_h\left(\frac{1}{2}t_1^2 + \frac{1}{3}\tilde{\theta}t_1^3\right) + \tilde{r}\tilde{C}_p(\tilde{\theta}t_1^2) + \frac{\tilde{r}}{2}\tilde{C}_s(T - t_1)^2\right]$$

$$= \frac{1}{T}\Big[C_o + (r_1C_{h1}, r_2C_{h2}, r_3C_{h3}, r_4C_{h4})\Big(\frac{t_1^2}{2} + \frac{\theta_1 t_1^3}{3}, \frac{t_1^2}{2} + \frac{\theta_2 t_1^3}{3}, \frac{t_1^2}{2} + \frac{\theta_3 t_1^3}{3}, \frac{t_1^2}{2} + \frac{\theta_4 t_1^3}{3}\Big) + (r_1C_{p1}, r_2C_{p2}, r_3C_{p3}, r_4C_{p4})(\theta_1 t_1^2, \theta_2 t_1^2, \theta_3 t_1^2, \theta_4 t_1^2) + (r_1C_{s1}, r_2C_{s2}, r_3C_{s3}, r_4C_{s4})\frac{(T-t_1)^2}{2}\Big]$$

$$= \frac{1}{T}\Big[C_o + \Big(r_1C_{h1}\Big(\frac{t_1^2}{2} + \frac{\theta_1 t_1^3}{3}\Big), r_2C_{h2}\Big(\frac{t_1^2}{2} + \frac{\theta_2 t_1^3}{3}\Big), r_3C_{h3}\Big(\frac{t_1^2}{2} + \frac{\theta_3 t_1^3}{3}\Big), r_4C_{h4}\Big(\frac{t_1^2}{2} + \frac{\theta_4 t_1^3}{3}\Big)\Big) + (r_1C_{p1}\theta_1 t_1^2, r_2C_{p2}\theta_2 t_1^2, r_3C_{p3}\theta_3 t_1^2, r_4C_{p4}\theta_4 t_1^2) + \Big(r_1C_{s1}\frac{(T-t_1)^2}{2}, r_2C_{s2}\frac{(T-t_1)^2}{2}, r_3C_{s3}\frac{(T-t_1)^2}{2}, r_4C_{s4}\frac{(T-t_1)^2}{2}\Big)\Big]$$

$$= (W, X, Y, Z), \qquad (15)$$

where $W = \frac{1}{T}\left(C_o + r_1C_{h1}\left(\frac{t_1^2}{2} + \frac{\theta_1 t_1^3}{3}\right) + r_1C_{p1}\theta_1 t_1^2 + r_1C_{s1}\frac{(T-t_1)^2}{2}\right)$,

$X = \frac{1}{T}\left(C_o + r_2C_{h2}\left(\frac{t_1^2}{2} + \frac{\theta_2 t_1^3}{3}\right) + r_2C_{p2}\theta_2 t_1^2 + r_2C_{s2}\frac{(T-t_1)^2}{2}\right)$,

$Y = \frac{1}{T}\left(C_o + r_3C_{h3}\left(\frac{t_1^2}{2} + \frac{\theta_3 t_1^3}{3}\right) + r_3C_{p3}\theta_3 t_1^2 + r_3C_{s3}\frac{(T-t_1)^2}{2}\right)$,

$Z = \frac{1}{T}\left(C_o + r_4C_{h4}\left(\frac{t_1^2}{2} + \frac{\theta_4 t_1^3}{3}\right) + r_4C_{p4}\theta_4 t_1^2 + r_4C_{s4}\frac{(T-t_1)^2}{2}\right)$.

The $\alpha -$ cuts, $C_L(\alpha)$ and $C_R(\alpha)$, of trapezoidal fuzzy number $\tilde{C}(t_1, T)$, are given

$C_L(\alpha) = W + (X - W)\alpha = \frac{1}{T}\left(C_o + r_1C_{h1}\left(\frac{t_1^2}{2} + \frac{\theta_1 t_1^3}{3}\right) + r_1C_{p1}\theta_1 t_1^2 + r_1C_{s1}\frac{(T-t_1)^2}{2}\right) + \frac{1}{T}\Big[r_2C_{h2}\left(\frac{t_1^2}{2} + \frac{\theta_2 t_1^3}{3}\right) - r_1C_{h1}\left(\frac{t_1^2}{2} + \frac{\theta_1 t_1^3}{3}\right) + r_2C_{p2}\theta_2 t_1^2 - r_1C_{p1}\theta_1 t_1^2 + r_2C_{s2}\frac{(T-t_1)^2}{2} - r_1C_{s1}\frac{(T-t_1)^2}{2}\Big]\alpha$,

And $C_R(\alpha) = Z - (Z - Y)\alpha = \frac{1}{T}\left(C_o + r_4C_{h4}\left(\frac{t_1^2}{2} + \frac{\theta_4 t_1^3}{3}\right) + r_4C_{p4}\theta_4 t_1^2 + r_4C_{s4}\frac{(T-t_1)^2}{2}\right) - \frac{1}{T}\Big[r_4C_{h4}\left(\frac{t_1^2}{2} + \frac{\theta_4 t_1^3}{3}\right) - r_3C_{h3}\left(\frac{t_1^2}{2} + \frac{\theta_3 t_1^3}{3}\right) + r_4C_{p4}\theta_4 t_1^2 - r_3C_{p3}\theta_3 t_1^2 + r_4C_{s4}\frac{(T-t_1)^2}{2} - r_3C_{s3}\frac{(T-t_1)^2}{2}\Big]\alpha$

By using signed distance method, the defuzzified value of fuzzy number $\tilde{C}(t_1, T)$, is given by

$$C_{ds}(t_1, T) = \frac{1}{2}\int_0^1 [C_L(\alpha) + C_R(\alpha)]d\alpha$$

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$= \frac{1}{2T}[C_o + r_1 C_{h1}(\frac{t_1^2}{2} + \frac{\theta_1 t_1^3}{3}) + r_1 C_{p1}\theta_1 t_1^2 + r_1 C_{s1}\frac{(T-t_1)^2}{2}] + \frac{1}{4T}[r_2 C_{h2}\left(\frac{t_1^2}{2} + \frac{\theta_2 t_1^3}{3}\right)$
$- r_1 C_{h1}(\frac{t_1^2}{2} + \frac{\theta_1 t_1^3}{3}) + r_2 C_{p2}\theta_2 t_1^2 - r_1 C_{p1}\theta_1 t_1^2 + r_2 C_{s2}\frac{(T-t_1)^2}{2} - r_1 C_{s1}\frac{(T-t_1)^2}{2}] + \frac{1}{2T}[C_o + r_4 C_{h4}(\frac{t_1^2}{2} + \frac{\theta_4 t_1^3}{3}) + r_4 C_{p4}\theta_4 t_1^2 + r_4 C_{s4}\frac{(T-t_1)^2}{2}] - \frac{1}{4T}[r_4 C_{h4}(\frac{t_1^2}{2} + \frac{\theta_4 t_1^3}{3})$
$- r_3 C_{h3}(\frac{t_1^2}{2} + \frac{\theta_3 t_1^3}{3}) + r_4 C_{p4}\theta_4 t_1^2 - r_3 C_{p3}\theta_3 t_1^2 + r_4 C_{s4}\frac{(T-t_1)^2}{2} - r_3 C_{s3}\frac{(T-t_1)^2}{2}]$ (16)

To minimize total cost function per unit time $C_{ds}(t_1, T)$, the optimal value of t_1 and T can be obtained by solving the following equations:

$\frac{\partial C_{ds}(t_1,T)}{\partial t_1} = 0$,

$\Rightarrow \frac{1}{2T}[r_1 C_{h1}(t_1+\theta_1 t_1^2)+2r_1 C_{p1}\theta_1 t_1 - r_1 C_{s1}(T-t_1)]$
$+ \frac{1}{4T}[r_2 C_{h2}(t_1+\theta_2 t_1^2) - r_1 C_{h1}(t_1+\theta_1 t_1^2) +$
$2r_2 C_{p2}\theta_2 t_1 - 2r_1 C_{p1}\theta_1 t_1 - r_2 C_{s2}(T-t_1) + r_1 C_{s1}(T-t_1)]$
$+ \frac{1}{2T}[r_4 C_{h4}(t_1+\theta_4 t_1^2)+2r_4 C_{p4}\theta_4 t_1 - r_4 C_{s4}(T-t_1)]$
$- \frac{1}{4T}[r_4 C_{h4}(t_1+\theta_4 t_1^2) - r_3 C_{h3}(t_1+\theta_3 t_1^2) +$
$2 r_4 C_{p4}\theta_4 t_1 - 2r_3 C_{p3}\theta_3 t_1 - r_4 C_{s4}(T-t_1) + r_3 C_{s3}(T-t_1)] = 0$

$\Rightarrow$
$2[r_1 C_{h1}(t_1+\theta_1 t_1^2)+2r_1 C_{p1}\theta_1 t_1 - r_1 C_{s1}(T-t_1)] + [r_2 C_{h2}(t_1+\theta_2 t_1^2) - r_1 C_{h1}(t_1+\theta_1 t_1^2) +$
$2r_2 C_{p2}\theta_2 t_1 - 2r_1 C_{p1}\theta_1 t_1 - r_2 C_{s2}(T-t_1) + r_1 C_{s1}(T-t_1)$
$] + 2[r_4 C_{h4}(t_1+\theta_4 t_1^2)+2r_4 C_{p4}\theta_4 t_1 - r_4 C_{s4}(T-t_1)]$
$- [r_4 C_{h4}(t_1+\theta_4 t_1^2) - r_3 C_{h3}(t_1+\theta_3 t_1^2) +$
$2 r_4 C_{p4}\theta_4 t_1 - 2r_3 C_{p3}\theta_3 t_1 - r_4 C_{s4}(T-t_1) + r_3 C_{s3}(T-t_1)] = 0$ (17)

and $\frac{\partial C_{ds}(t_1,T)}{\partial T} = -\frac{1}{2T^2}[C_o + r_1 C_{h1}(\frac{t_1^2}{2} + \frac{\theta_1 t_1^3}{3}) + r_1 C_{p1}\theta_1 t_1^2 + r_1 C_{s1}\frac{(T-t_1)^2}{2}] + \frac{1}{2T}[r_1 C_{s1}(T-t_1)] -$
$\frac{1}{4T^2}[r_2 C_{h2}\left(\frac{t_1^2}{2} + \frac{\theta_2 t_1^3}{3}\right) - r_1 C_{h1}(\frac{t_1^2}{2} + \frac{\theta_1 t_1^3}{3}) +$
$r_2 C_{p2}\theta_2 t_1^2 - r_1 C_{p1}\theta_1 t_1^2 + r_2 C_{s2}\frac{(T-t_1)^2}{2} - r_1 C_{s1}\frac{(T-t_1)^2}{2}] +$
$\frac{1}{4T}[r_2 C_{s2}(T-t_1) - r_1 C_{s1}(T-t_1)] - \frac{1}{2T^2}[C_o + r_4 C_{h4}(\frac{t_1^2}{2} + \frac{\theta_4 t_1^3}{3})$
$+ r_4 C_{p4}\theta_4 t_1^2 + r_4 C_{s4}\frac{(T-t_1)^2}{2}] + \frac{1}{2T}[r_4 C_{s4}(T-t_1)] +$
$\frac{1}{4T^2}[r_4 C_{h4}(\frac{t_1^2}{2} + \frac{\theta_4 t_1^3}{3}) - r_3 C_{h3}(\frac{t_1^2}{2} + \frac{\theta_3 t_1^3}{3}) +$
$r_4 C_{p4}\theta_4 t_1^2 - r_3 C_{p3}\theta_3 t_1^2 + r_4 C_{s4}\frac{(T-t_1)^2}{2} - r_3 C_{s3}\frac{(T-t_1)^2}{2}]$
$- \frac{1}{4T}[r_4 C_{s4}(T-t_1) - r_3 C_{s3}(T-t_1)] = 0$

$\Rightarrow 4C_o + 2[r_1 C_{h1}(\frac{t_1^2}{2} + \frac{\theta_1 t_1^3}{3}) + r_1 C_{p1}\theta_1 t_1^2 + r_1 C_{s1}\frac{(T-t_1)^2}{2}]$
$-2T[r_1 C_{s1}(T-t_1)] + [r_2 C_{h2}\left(\frac{t_1^2}{2} + \frac{\theta_2 t_1^3}{3}\right) - r_1 C_{h1}\left(\frac{t_1^2}{2} + \frac{\theta_1 t_1^3}{3}\right) +$
$r_2 C_{p2}\theta_2 t_1^2 - r_1 C_{p1}\theta_1 t_1^2 + r_2 C_{s2}\frac{(T-t_1)^2}{2} - r_1 C_{s1}\frac{(T-t_1)^2}{2}] - T[$
$r_2 C_{s2}(T-t_1) - r_1 C_{s1}(T-t_1)] + 2[r_4 C_{h4}(\frac{t_1^2}{2} + \frac{\theta_4 t_1^3}{3}) +$
$r_4 C_{p4}\theta_4 t_1^2 + r_4 C_{s4}\frac{(T-t_1)^2}{2}] -$
$2T[r_4 C_{s4}(T-t_1)] - [r_4 C_{h4}(\frac{t_1^2}{2} + \frac{\theta_4 t_1^3}{3}) - r_3 C_{h3}(\frac{t_1^2}{2} + \frac{\theta_3 t_1^3}{3}) +$
$r_4 C_{p4}\theta_4 t_1^2 - r_3 C_{p3}\theta_3 t_1^2 +$
$r_4 C_{s4}\frac{(T-t_1)^2}{2} - r_3 C_{s3}\frac{(T-t_1)^2}{2}] + T[r_4 C_{s4}(T-t_1)$
$- r_3 C_{s3}(T-t_1)] = 0$

$\Rightarrow 4C_o + 2[r_1 C_{h1}(\frac{t_1^2}{2} + \frac{\theta_1 t_1^3}{3}) + r_1 C_{p1}\theta_1 t_1^2 + r_1 C_{s1}\frac{(T-t_1)^2}{2}] +$
$[r_2 C_{h2}\left(\frac{t_1^2}{2} + \frac{\theta_2 t_1^3}{3}\right) - r_1 C_{h1}\left(\frac{t_1^2}{2} + \frac{\theta_1 t_1^3}{3}\right) + r_2 C_{p2}\theta_2 t_1^2$
$- r_1 C_{p1}\theta_1 t_1^2 + r_2 C_{s2}\frac{(T-t_1)^2}{2} - r_1 C_{s1}\frac{(T-t_1)^2}{2}] - T(T-t_1)[$
$r_1 C_{s1} + r_2 C_{s2} + r_3 C_{s3} + r_4 C_{s4}] + 2[r_4 C_{h4}(\frac{t_1^2}{2} + \frac{\theta_4 t_1^3}{3}) +$
$r_4 C_{p4}\theta_4 t_1^2 +$
$r_4 C_{s4}\frac{(T-t_1)^2}{2}] - [r_4 C_{h4}(\frac{t_1^2}{2} + \frac{\theta_4 t_1^3}{3}) - r_3 C_{h3}(\frac{t_1^2}{2} + \frac{\theta_3 t_1^3}{3}) +$
$r_4 C_{p4}\theta_4 t_1^2 - r_3 C_{p3}\theta_3 t_1^2 + r_4 C_{s4}\frac{(T-t_1)^2}{2} - r_3 C_{s3}\frac{(T-t_1)^2}{2}] = 0$ (18)

We solve the non linear equations (17) and (18) by using computer software MATHEMATICA 8.0. But, to find out the second derivatives of the total cost function is very difficult and lengthy. However, with the help of graph, we can easily demonstrate the convexity of total fuzzy cost function $C_{ds}(t_1, T)$, (see Figure 3). Similarly, the total shortage per unit time in fuzzy sense is given by

$\tilde{C}_{TS} = \frac{r}{2T}\tilde{C}_s(T-t_1)^2$
$= \frac{(T-t_1)^2}{2T}(r_1 C_{s1}, r_2 C_{s2}, r_3 C_{s3}, r_4 C_{s4})$ (19)

Defuzzified value of fuzzy number $\tilde{C}_{TS}$ by using Signed distance method, is given by

$(C_{TS})_{ds} = \frac{(T-t_1)^2}{8T}[r_1 C_{s1} + r_2 C_{s2} + r_3 C_{s3} + r_4 C_{s4}]$ (20)

VI. NUMERICAL EXAMPLE

To illustrate the proposed method, let us consider the following input data:

Crisp Model

Input data: C_o= Rs 200 per order, C_h= Rs 5 per unit per year, C_s= Rs15 per unit per year, C_p= Rs 20 per unit per year, θ = 0.01per year, r = 110 per year. The solution of crisp model is: t_1= 0.7002 year, T = 0.9539 year, C_{TS} = 55.6663, and $C(t_1, T)$ = Rs 418.642.

To show the convexity of cost function $C(t_1, T)$, we plot a 3D graph among t_1 and T, where values of both t_1 and T ranging from 0.5000 to 1.000. A three-dimensional graph is shown in the following Figure 2.

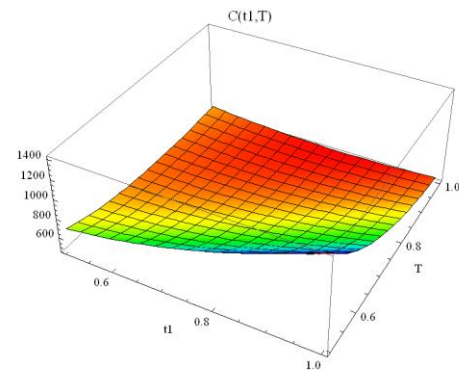

Figure 2: Convexity of Cost Function with respect to t_1 and T, (Crisp Model).

Fuzzy Model

Let $\tilde{C}_h$ = (2, 4, 6, 8), $\tilde{C}_p$ = (14, 18, 22, 26), $\tilde{C}_s$ = (12,14, 16, 18), $\tilde{r}$= (80, 100, 120, 140), $\tilde{\theta}$ = (0.004, 0.008, 0.012, 0.016). Then, by using Signed Distance Method, we obtain:

CASE–I: When $\tilde{C}_h$, $\tilde{C}_p$, $\tilde{C}_s$, $\tilde{r}$, and $\tilde{\theta}$ are fuzzy trapezoidal numbers.

The solution of fuzzy model is: t_1= 0.6605 year, T = 0.9167 year, $(C_{TS})_{ds}$ = 60.8625, and $C_{ds}(t_1, T)$ = Rs 435.313.

CASE–II: When $\tilde{C}_p, \tilde{C}_s$, $\tilde{r}$, and $\tilde{\theta}$ are fuzzy trapezoidal numbers.

The solution of fuzzy model is: t_1= 0.6951 year, T = 0.9439 year, $(C_{TS})_{ds}$ = 55.7434, and $C_{ds}(t_1, T)$ = Rs 422.886.

CASE–III: When $\tilde{C}_s$, $\tilde{r}$, and $\tilde{\theta}$ are fuzzy trapezoidal numbers.

The solution of fuzzy model is: t_1= 0.6997 year, T = 0.9476 year, $(C_{TS})_{ds}$ = 55.1248, and $C_{ds}(t_1, T)$ = Rs 422.866.

CASE–IV: When $\tilde{r}$, and $\tilde{\theta}$ are fuzzy trapezoidal numbers.

The solution of fuzzy model is: t_1= 0.6970 year, T = 0.9513 year, $(C_{TS})_{ds}$ = 56.0827, and $C_{ds}(t_1, T)$ = Rs 419.560.

CASE–V: When $\tilde{r}$ is a fuzzy trapezoidal number.

The solution of fuzzy model is: t_1= 0.7002 year, T = 0.9539 year, $(C_{TS})_{ds}$ = 55.6663, and $C_{ds}(t_1, T)$ = Rs 418.726.

CASE-VI: When none of $\tilde{C}_h$, $\tilde{C}_p, \tilde{C}_s$, $\tilde{r}$, and $\tilde{\theta}$ is a fuzzy trapezoidal number.

The solution of fuzzy model is: t_1= **0.7002** year, T = **0.9539** year, $(C_{TS})_{ds}$ = 55.6663, and $C_{ds}(t_1, T)$ = Rs **418.642**.

Table1. Comparison of Optimal Results

Model	Optimal value of t_1, (year)	Optimal value of *T*, (year)	Optimal value of C_{TS}, (Rs.)	Optimal value of C(t_1 ,T), (Rs.)
Crisp Model	0.7002	0.9539	55.6663	418.642
Fuzzy Model	0.6605	0.9167	60.8625	435.313

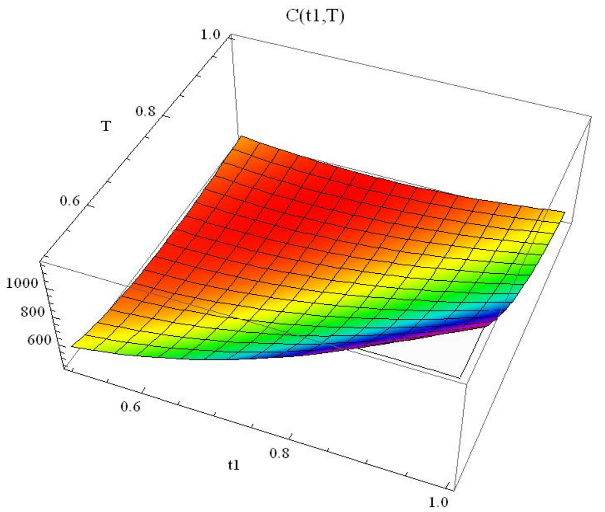

Figure 3: Convexity of Cost Function with respect to t_1 and T, (Fuzzy Model).

VII. SENSITIVITY ANALYSIS

To study the effects of changes in the system parameters, the sensitivity is analyzed. The results are shown in below tables.

Table 2. Sensitivity Analysis on parameter *r*

Defuzzify value of *r*, *(units/year)*	Fuzzify value of parameter *r*	t_1 (year)	T (year)	$C_{ds}(t_1, T)$ (Rs.)
90	(60, 80, 100, 120)	0.7216	1.005	396.838
100	(70, 90, 110 , 130)	0.6890	0.9579	416.527
110	(80, 100, 120, 140)	0.6605	0.9167	435.313
120	(90, 110, 130, 150)	0.6352	0.8803	453.319
130	(100, 120, 140, 160)	0.6126	0.8480	470.634

Table 3. Sensitivity Analysis on parameter θ

Defuzzify value of θ, *(per year)*	Fuzzify value of parameter θ	t_1 (year)	T (year)	$C_{ds}(t_1, T)$ (Rs.)
0.006	(0, 0.004, 0.008, 0.012)	0.6745	0.9283	430.603
0.008	(0.002, 0.006, 0.010, 0.014)	0.6678	0.9230	432.976
0.010	(0.004, 0.008, 0.012, 0.016)	0.6605	0.9167	435.313
0.012	(0.006, 0.010, 0.014, 0.018)	0.6541	0.9116	437.617
0.014	(0.008, 0.012, 0.016, 0.020)	0.6479	0.9068	439.888

Table 4. Sensitivity Analysis on parameter C_s

Defuzzify value of C_s, (Rs per unit per year)	Fuzzify value of parameter C_s	t_1 (year)	T (year)	$C_{ds}(t_1, T)$ (Rs.)
11	(8, 10, 12, 14)	0.6181	0.9512	415.612
13	(10, 12, 14, 16)	0.6347	0.9259	426.631
15	(12, 14, 16, 18)	0.6605	0.9167	435.313
17	(14, 16, 18, 20)	0.6712	0.9018	442.444
19	(16, 18, 20, 22)	0.6831	0.8937	448.368

Table 5. Sensitivity Analysis on parameter C_p

Defuzzify value of C_p, (Rs per unit per year)	Fuzzify value of parameter C_p	t_1 (year)	T (year)	$C_{ds}(t_1, T)$ (Rs.)
16	(10, 14, 18, 22)	0.6666	0.9215	433.009
18	(12, 16, 20, 24)	0.6635	0.9191	434.165
20	(14, 18, 22, 26)	0.6605	0.9167	435.313
22	(16, 20, 24, 28)	0.6574	0.9143	436.455
24	(18, 22, 26, 30)	0.6544	0.9120	437.588

Table 6. Sensitivity Analysis on parameter C_h

Defuzzify value of C_h, (Rs per unit per year)	Fuzzify value of parameter C_h	t_1 (year)	T (year)	$C_{ds}(t_1, T)$ (Rs.)
3	(0, 2, 4, 6)	0.8500	1.0694	372.698
4	(1, 3, 5, 7)	0.7411	0.9806	406.750
5	(2, 4, 6, 8)	0.6605	0.9167	435.313
6	(3, 5, 7, 9)	0.5976	0.8682	459.817
7	(4, 6, 8, 10)	0.5469	0.8300	481.063

VIII. OBSERVATIONS

1) From Table 2, as we increase the demand rate r, the optimum values of t_1 and T decrease. By this effect, the total cost increases.
2) From Table 3, as we increase the deterioration rate θ, the optimum values of t_1 and T decrease. By this effect, the total cost increases.
3) From Table 4, as we increase the shortage cost C_s, the optimum value of $t_1$increases, and optimum value of T decreases, and finally the total cost increases.

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4) From Table 5, as we increase the purchasing cost C_p, the optimum values of t_1 and T decrease. By this effect, the total cost increases.
5) From Table 6, as we increase the holding cost C_h, the optimum values of t_1 and T decrease. By this effect, the total cost increases.
6) Total cost function is more sensitive to the changes in holding cost, and in demand rate (Table 6&2), while it is less sensitive to the changes in purchasing cost, and in deterioration (Table 5&3).
7) In comparing the optimal results obtained in crisp model and in fuzzy model, we observe that the optimal values of shortages cost and total cost increase in fuzzy model (Table 1).
8) In CASE-VI, when none of $\tilde{C}_h$, $\tilde{C}_p, \tilde{C}_s$, $\tilde{r}$, and $\tilde{\theta}$ is a fuzzy trapezoidal number, the solution of fuzzy model is the same as that of crisp model. So, when we change the nature of all the parameters - $\tilde{C}_h$, $\tilde{C}_p, \tilde{C}_s$, $\tilde{r}$, and $\tilde{\theta}$ from fuzziness to crispness, the concerned fuzzy model immediately becomes the crisp model.

IX. CONCLUSIONS

This paper presented a fuzzy inventory model for deteriorating items with shortages under fully backlogged condition. Shortages and deterioration are natural in any inventory control system. Demand rate and deterioration rate were both assumed to be constant. The proposed model is developed in both the crisp and fuzzy environments. In fuzzy environment, all related inventory parameters were assumed to be trapezoidal fuzzy numbers. The optimum results of fuzzy model are defuzzified by Signed distance method. Sensitivity analysis indicates that the total cost function is more sensitive to the changes in the holding cost. So, the decision maker, after analyzing the result, can plan for the optimal value for total cost, and for other related parameters.

REFERENCES

[1] F. Harris. Operations and cost, AW Shaw Co. Chicago, 1915.
[2] L. A. Zadeh, "Fuzzy sets", Information Control, vol. 8, 1965, pp.338-353.
[3] L. A. Zadeh and R. E. Bellman, "Decision making in a fuzzy environment", Management Science, vol. 17, 1970, pp.140-164.
[4] R. Jain, "Decision making in the presence of fuzzy variables", IIIE Transactions on systems, Man and Cybernetics, vol. 17, 1976, pp.698-703.
[5] D. Dubois and H. Prade, "Operations on fuzzy numbers", International Journal of Systems Science, vol. 9(6), 1978, pp.613-626.
[6] J. Kacpryzk and P. Staniewski, "Long-term inventory policy-making through fuzzy-decision making models", Fuzzy Sets and Systems, vol. 8, 1982, pp.117-132.
[7] H. J. Zimmerman, "Using fuzzy sets in operational research", European Journal of Operational Research, vol. 13, 1983, pp.201-206.
[8] G. Urgeletti Tinarelli, "Inventory control models and problems", European Journal of Operational Research, vol. 14, 1983, pp.1-12.
[9] K. S. Park, "Fuzzy set theoretical interpretation of economic order quantity", IEEE Trans. Systems Man. Cybernet SMC, vol. 17, 1987, pp.1082-1084.
[10] M. Vujosevic, D. Petrovic and R. Petrovic, "EOQ formula when inventory cost is fuzzy", International Journal of Production Economics, vol. 45, 1996, pp.499-504
[11] J. S. Yao and H. M. Lee, "Fuzzy inventory with or without backorder for fuzzy order quantity with trapezoidal fuzzy number", Fuzzy sets and systems, vol. 105, 1999, pp.311-337.
[12] J. S. Yao and H. M. Lee, "Economic order quantity in fuzzy sense for inventory without backorder model", Fuzzy Sets and Systems, vol. 105, 1999, pp.13-31.
[13] C. K. Kao and W. K. Hsu, "A single-period inventory model with fuzzy demand", Computers and Mathematics with Applications, vol. 43, 2002, pp.841-848.
[14] C. H. Hsieh, "Optimization of fuzzy production inventory models", Information Sciences, vol. 146(1-4), 2002, pp.29-40.
[15] J. S. Yao and J. Chiang, "Inventory without back order with fuzzy total cost and fuzzy storing cost defuzzified by centroid and signed distance", European Journal of Operational Research, vol. 148, 2003, pp.401-409.
[16] Sujit De Kumar, P. K. Kundu and A. Goswami, "An economic production quantity inventory model involving fuzzy demand rate and fuzzy deterioration rate". Journal of Applied Mathematics and Computing, vol. 12(1-2), 2003, pp.251-260.
[17] J. K. Syed and L. A. Aziz, "Fuzzy inventory model without shortages using signed distance method", Applied Mathematics & Information Sciences, vol. 1(2), 2007, pp.203-209
[18] P. K. De and A. Rawat, "A fuzzy inventory model without shortages using triangular fuzzy number", Fuzzy Information & Engineering, vol. 1, 2011, pp.59-68
[19] C. K. Jaggi, S. Pareek, A. Sharma and Nidhi, "Fuzzy inventory model for deteriorating items with time-varying demand and shortages", American Journal of Operational Research, vol. 2(6), 2012, pp.81-92.
[20] Sumana Saha and Tripti Chakrabarti, "Fuzzy EOQ model for time dependent deteriorating items and time dependent demand with shortages", IOSR-Journal of Mathematics, vol. 2(4), Sept-Oct 2012, pp.46-54.
[21] D. Dutta and Pavan Kumar, "Fuzzy inventory model without shortage using trapezoidal fuzzy number with sensitivity analysis", IOSR-Journal of Mathematics, vol. 4(3), Nov-Dec 2012, pp.32-37. CrossRef, **doi:** 10.9790/5728-0433237
[22] D. Dutta and Pavan Kumar, "Optimal policy for an inventory model without shortages considering fuzziness in demand, holding cost and ordering cost", International Journal of Advanced Innovation and Research, vol. 2(3), 2013, pp.320-325.

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