Application of Sumudu Decomposition Method for Solving Linear and Nonlinear Klein-Gordon Equations

A. Ramadan, S. Al-luhaibi

Abstract In this paper, Sumudu decomposition method is applied to solve various forms of linear and nonlinear Klein-Gordon equations. The technique is a combined form of the Sumudu transform method and the Adomian decomposition method. The nonlinear term can easily be handled with the help of Adomian polynomials which is considered to be a clear advantage of this technique. We illustrate this technique with the help of four examples. The results reveal that the proposed algorithm is very efficient, simple and can be applied to other nonlinear problems.

***Keywords: Sumudu decomposition method; Sumudu transform; Adomian polynomials; Linear and nonlinear Klein-Gordon equations*.**

I. INTRODUCTION

The Klein-Gordon equations appear in quantum field theory, relativistic physics, dispersive wave-phenomena, plasma physics , nonlinear optics and applied and physical sciences, which are of the form[1,2]:

$$U_{tt}(x,t)-U_{xx}(x,t)+aUx,t)=g(x,t)\,, \qquad (1)$$

with the initial conditions

$$U(x,0)=h(x), \quad U_t(x,0)=f(x)\,. \qquad (2)$$

Nonlinear phenomena have important effects on applied mathematics, physics and related to engineering; many such physical phenomena are modeled in terms of nonlinear partial differential equations as in, Eq. (1).

The importance of obtaining the exact or approximate solutions of nonlinear partial differential equations in mathematics, physics and engineering is still a significant problem that needs new methods to discover exact or approximate solutions. Various powerful mathematical methods such as variational iteration method [3], homotopy perturbation method [4], new iterative method [5] and Adomian decomposition method [6] this method has proven useful for solving algebraic, differential, integro-differential, differential-delay and partial differential equations. In the nonlinear case for ordinary differential equations and partial differential equations, the method has the advantage of dealing directly with the problem [7, 8]. These equations are solved without transforming them to more simple ones. The method avoids linearization, perturbation, discretization, or any unrealistic assumptions [9, 10].

It was suggested in [11] that the noise terms appears always for inhomogeneous equations. Most recently, Wazwaz [12] established a necessary condition that is essentially needed to ensure the appearance of "noise terms" in the inhomogeneous equations.

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A. Ramadan, Department of Mathematics, Faculty of Science, Menoufia University, Egypt.

S. Al-luhaibi Department of Mathematics, Faculty of Science, Kirkuk University, Iraq.

In the present paper, application of the Sumudu decomposition method to solve the linear and nonlinear Klein-Gordon equations.

II. SUMUDU TRANSFORM

In the early 90's, in [13] introduced a new integral transform, named the Sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems. The Sumudu transform, is defined over the set of functions.

$$A=\left\{ f(t) \;\middle|\; \exists M,\tau_1,\tau_2>0, |f(t)|<Me^{\left(\frac{|t|}{\tau_j}\right)},\ if\quad t\in(-1)^j\times[0,\infty) \right\} \qquad (3)$$

by the following formula

$$\bar{f}(u)=S[f(t)]=\int_0^\infty f(ut)e^{-t}dt, \qquad u\in(-\tau_1,\tau_2)\,. \qquad (4)$$

Some of the properties were established in [14, 15]. In [16], further fundamental properties of this transform were also established. Similarly, this transform was applied to the one-dimensional neutron transport equation in [17]. In fact it was shown that there is a strong relationship between Sumudu and other integral transforms; see [18]. In particular the relation between Sumudu transform and Laplace transforms was proved in [19].

Further, in [20], the Sumudu transform was extended to the distributions and some of their properties were also studied in [21]. Recently, this transform is applied to solve the system of differential equations; see [22].

Note that a very interesting fact about Sumudu transform is that the original function and its Sumudu transform have the same Taylor coefficients except the factor n; see [23]. Thus if $f(t)=\sum_{n=0}^{\infty}a_n t^n$ then $F(u)=\sum_{n=0}^{\infty}n!a_n u^n$, see [18]. Similarly, the Sumudu transform sends combinations, $c(m,n)$, into permutations, $p(m,n)$ and hence it will be useful in the discrete systems.

III. BASIC IDEA OF SUMUDU DECOMPOSTION METHOD (SDM)

To Illustrate the basic idea of this method, we consider a general nonlinear non-homogeneous partial differential equation [24, 25. 26]:

$$DU(x,t)+RU(x,t)+NU(x,t)=g(x,t) \qquad (5)$$

$$U(x,0)=h(x), \qquad U_t(x,0)=f(x),$$

where D is the second order linear differential operator $D=\frac{\partial^2}{\partial t^2}$, R is the linear differential operator of less order than D, N represents the general nonlinear differential operator and $g(x,t)$ is the source term. Taking the Sumudu transform (denoted throughout this paper by S) on both sides of Eq. (5), we get:

$$S[DU(x,t)]=S[RU(x,t)]+S[NU(x,t)]=S[g(x,t)]. \quad (6)$$

Using the differentiation property of the Sumudu transform and above initial conditions, we have

$$S[DU(x,t)]=Su^2[g(x,t)]+h(x)+uf(x)-Su^2[RU(x,t)+NU(x,t)] \quad (7)$$

Now, applying the inverse Sumudu transform on both sides of Eq. (3.3), we get

$$U(x,t)=G(x,t)-S^{-1}\left[Su^2[RU(x,t)+NU(x,t)]\right], \quad (8)$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. The second step in Sumudu decomposition method is that we represent solution as an infinite series given below

$$U(x,t)=\sum_{n=0}^{\infty}U_n(x,t), \quad (9)$$

and the nonlinear term can be decomposed as:

$$NU(x,t)=\sum_{n=0}^{\infty}A_n, \quad (10)$$

where A_n are Adomian polynomials [27] of $U_0,U_1,U_2,\dots,U_n$ and it can be calculated by formula

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{0}^{\infty}\lambda^i U_i\right)\right]_{\lambda=0}, \quad n=0,1,2,\dots . \quad (11)$$

Using Eq. (9) and Eq. (10) in Eq. (8), we get

$$\sum_{n=0}^{\infty}U(x,t)=G(x,t)-S^{-1}\left[Su^2\left[R\sum_{n=0}^{\infty}U_n(x,t)+\sum_{n=0}^{\infty}A_n\right]\right] \quad (12)$$

On comparing both sides of the Eq. (12), we get

$$\begin{aligned}
&U_0(x,t)=G(x,t),\\
&U_1(x,t)=-S^{-1}\left[Su^2[RU_0(x,t)+A_0]\right],\\
&U_2(x,t)=-S^{-1}\left[Su^2[RU_1(x,t)+A_1]\right],\\
&U_3(x,t)=-S^{-1}\left[Su^2[RU_2(x,t)+A_2]\right].\\
&\vdots
\end{aligned} \quad (13)$$

In general the recursive relation is given by

$$U_0(x,t)=G(x,t),$$

$$U_{n+1}(x,t)=-S^{-1}\left[Su^2[RU_n(x,t)+A_n]\right], \quad n\ge 0. \quad (14)$$

Now first of all applying the Sumudu transform of the right hand side of Eq. (14) then applying the inverse Sumudu transform, we get the values of $U_0,U_1,U_2,\dots,U_n$ respectively.

IV. NUMERICAL APPLICATIONS

In this section, we use SDM in solving the linear and nonlinear Klein-Gordon equations.

Example 4.1

Consider the following linear Klein-Gordon equation [28, 29, 30]:

$$U_{tt}(x,t)-U_{xx}(x,t)+U(x,t)=0, \quad (15)$$

with the initial conditions

$$U(x,0)=0,\ U_t(x,0)=x.$$

By taking Sumudu transform for (15), we obtain

$$S[U(x,t)]=ux+u^2S[U_{xx}(x,t)-Ux,t)]. \quad (16)$$

By applying the inverse Sumudu transform for (16), we get

$$U(x,t)=xt+S^{-1}\left[u^2S[U_{xx}(x,t)-Ux,t)]\right]. \quad (17)$$

Following the technique, if we assume an infinite series solution of the form (9), we obtain

$$\sum_{n=0}^{\infty}U_{n+1}(x,t)=xt+S^{-1}\left[u^2S\left[\sum_{n=0}^{\infty}(U_n)_{xx}(x,t)-\sum_{n=0}^{\infty}(U_n)(x,t)\right]\right] \quad (18)$$

Note that, Eq. (15), don't have nonlinear term, then $A_n=0$.

From the relationship in (14), we obtain

$$U_0(x,t)=G(x,t)=xt,$$

$$U_1(x,t)=-S^{-1}\left[Su^2[(U_0)_{xx}(x,t)+(U_0)(x,t)]\right]=-\frac{xt^3}{6},$$

$$U_2(x,t)=-S^{-1}\left[Su^2[(U_1)_{xx}(x,t)+(U_1)(x,t)]\right]=\frac{xt^5}{120},$$

$$(19)\ U_3(x,t)=-S^{-1}\left[Su^2[(U_2)_{xx}(x,t)+(U_2)(x,t)]\right]=-\frac{xt^7}{5040},$$

$$\vdots$$

which in closed form gives exact solution

$$U(x,t)=\sum_{i=0}^{\infty}U_i(x,t)=x\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\dots\right)=x\sin(t), \quad (20)$$

which is the same result obtained by homotopy perturbation transform method [29], and variational iteration method [30].

Example 4.2

Consider the following linear Klein-Gordon equation [28, 29, 30]:

$$U_{tt}(x,t)-U_{xx}(x,t)+U(x,t)=2\sin x, \quad (21)$$

with the initial conditions

$$U(x,0)=\sin x,\ U_t(x,0)=1.$$

By taking Sumudu transform for (21), we obtain

$$S[U(x,t)]=\sin x+u+2u^2\sin x+u^2S[U_{xx}(x,t)-U(x,t)] \quad (22)$$

By applying the inverse Sumudu transform for (22), we get

$$U(x,t)=t^2\sin x+\sin x+t+S^{-1}\left[u^2S[U_{xx}(x,t)-Ux,t)]\right] \quad (23)$$

Following the technique, if we assume an infinite series solution of the form (9), we obtain

$$\sum_{n=0}^{\infty}U(x,t)=t^2\sin x+\sin x+t+S^{-1}\left[u^2S\left[\sum_{n=0}^{\infty}(U_n)_{xx}(x,t)-\sum_{n=0}^{\infty}(U_n)(x,t)\right]\right]. \quad (24)$$

Note that, Eq. (21), don't have nonlinear term, then $A_n=0$.

From the relationship in (14), we obtain

$$U_0(x,t)=G(x,t)=t^2\sin x+\sin x+t,$$

$$U_1(x,t)=-S^{-1}\left[Su^2[(U_0)_{xx}(x,t)+(U_0)(x,t)]\right]=-t^2\sin x-\frac{t^3}{3!}-\frac{t^4}{3!}\sin x$$

,

$$U_2(x,t) = -S^{-1}\left[Su^2\left[(U_1)_{xx}(x,t) + (U_1)(x,t)\right]\right]$$
$$= \frac{t^6}{90}\sin x + \frac{t^5}{5!} + \frac{t^4}{3!}\sin x \quad , \tag{25}$$
$$U_3(x,t) = -S^{-1}\left[Su^2\left[(U_2)_{xx}(x,t) + (U_2)(x,t)\right]\right]$$
$$= \frac{t^7}{7!} - \frac{t^6}{90}\sin x - \frac{2t^8}{7!}\sin x$$
$$\vdots$$

which in closed form gives exact solution

$$U(x,t) = \sum_{i=0}^{\infty} U_i(x,t) = \sin x + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots\right), \tag{26}$$
$$= \sin x + \sin t$$

which is the same result obtained by homotopy perturbation transform method [29], and variational iteration method [30].

Example 4.3

Consider the following nonlinear Klein-Gordon equation [28, 29, 30]:

$$U_{tt}(x,t) - U_{xx}(x,t) + U^2(x,t) = x^2t^2, \tag{27}$$

with the initial conditions

$$U(x,0) = 0, \quad U_t(x,0) = x.$$

By taking Sumudu transform for (27), we obtain

$$S[U(x,t)] = ux + 2x^2u^4 + u^2S\left[U_{xx}(x,t) - U^2(x,t)\right]. \tag{28}$$

By applying the inverse Sumudu transform for (28) , we get

$$U(x,t) = xt + \frac{x^2t^4}{12} + S^{-1}\left[u^2S\left[U_{xx}(x,t) - U^2(x,t)\right]\right]. \tag{29}$$

Following the technique, if we assume an infinite series solution of the form (9) and (10), we obtain

$$\sum_{n=0}^{\infty} U(x,t) = xt + \frac{x^2t^4}{12}$$
$$+ S^{-1}\left[u^2S\left[\sum_{n=0}^{\infty}(U_n)_{xx}(x,t) - \sum_{n=0}^{\infty}A_n(U)\right]\right]. \tag{30}$$

In (30), $A_n(U)$ are Adomian polynomials that represent nonlinear term. So Adomian polynomials are given as follows:

$$A_n(U) = U^2(x,t). \tag{31}$$

The few components of the Adomian polynomials are given as follows:

$$A_0 = U_0^2,$$
$$A_1 = 2U_0U_1, \tag{32}$$
$$A_2 = 2U_0U_2 + U_1^2,$$
$$\vdots$$

From the relationship in (14), we obtain

$$U_0(x,t) = G(x,t) = xt + \frac{x^2t^4}{12},$$
$$U_1(x,t) = -S^{-1}\left[Su^2\left[(U_0)_{xx}(x,t) + A_0(U)\right]\right]$$
$$= \frac{t^6}{180} - \frac{x^4t^{10}}{12960} - \frac{x^3t^7}{252} - \frac{x^2t^4}{12} \quad ,$$

$$U_2(x,t) = -S^{-1}\left[Su^2\left[(U_1)_{xx}(x,t) + A_1(U)\right]\right]$$
$$= \frac{x^2t^{12}}{71280} - \frac{xt^9}{22680} - \frac{t^6}{180} + \frac{x^6t^{16}}{18662400} \quad , \tag{33}$$
$$- \frac{11x^4t^{10}}{45360} - \frac{383x^5t^{13}}{15921360} + \frac{x^3t^7}{252}$$
$$\vdots$$

which in closed form gives exact solution

$$U(x,t) = \sum_{i=0}^{\infty} U_i(x,t) = xt, \tag{34}$$

which is the same result obtained by homotopy perturbation transform method [29], and variational iteration method [30].

Example 4.4

Consider the following nonlinear Klein-Gordon equation, [28, 29, 30]:

$$U_{tt}(x,t) - U_{xx}(x,t) + U^2(x,t) = 2x^2 - 2t^2 + x^4t^4, \tag{35}$$

with the initial conditions

$$U(x,0) = 0, \quad U_t(x,0) = 0.$$

By taking Sumudu transform for (35), we obtain

$$S[U(x,t)] = 2x^2u^2 - 4u^4 + 24x^4u^6$$
$$+ u^2S\left[U_{xx}(x,t) - U^2(x,t)\right]. \tag{36}$$

By applying the inverse Sumudu transform for (36), we get

$$U(x,t) = x^2t^2 - \frac{t^4}{6} + \frac{x^4t^6}{30}$$
$$+ S^{-1}\left[u^2S\left[U_{xx}(x,t) - U^2(x,t)\right]\right]. \tag{37}$$

Following the technique, if we assume an infinite series solution of the form (9) and (10), we obtain

$$\sum_{n=0}^{\infty} U_{n+1}(x,t) = x^2t^2 - \frac{t^4}{6} + \frac{x^4t^6}{30}$$
$$+ S^{-1}\left[u^2S\left[\sum_{n=0}^{\infty}(U_n)_{xx}(x,t) - \sum_{n=0}^{\infty}A_n(U)\right]\right] \tag{38}$$

In (38), $A_n(U)$ are Adomian polynomials that represent nonlinear term. So Adomian polynomials are given as follows:

$$\sum_{n=0}^{\infty} A_n(U) = U^2(x,t). \tag{39}$$

The few components of the Adomian polynomials are given as follows:

$$A_0 = U_0^2,$$
$$A_1 = 2U_0U_1, \tag{40}$$
$$A_2 = 2U_0U_2 + U_1^2,$$
$$\vdots$$

From the relationship in (14), we obtain

$$U_0(x,t) = G(x,t) = x^2t^2 - \frac{t^4}{6} + \frac{x^4t^6}{30},$$

$$U_1(x,t) = -S^{-1}\left[Su^2\left[(U_0)_{xx}(x,t) + A_0(U)\right]\right]$$
$$= -\frac{x^8t^{14}}{163800} + \frac{x^4t^{12}}{11880} - \frac{x^6t^{10}}{1350} - \frac{x^6t^{16}}{18662400}, \quad (41)$$
$$-\frac{t^{10}}{3240} + \frac{11x^2t^8}{840} - \frac{x^4t^6}{30} + \frac{t^6}{6}$$

⋮

In a similar manner as above the solution U(x, t) is given by

$$U(x,t) = \sum_{i=0}^{\infty} U_i(x,t) = x^2t^2, \quad (42)$$

Which is the same result obtained by homotopy perturbation transform method [29], and variational iteration method [30].

V. CONCLUSION

In this paper, the SDM has been applied to linear and nonlinear Klein-Gordon equations with initial conditions. Four examples have been presented. The results show that the SDM is powerful and efficient technique in finding exact and approximate solutions for nonlinear differential equations amounts to an improvement of the performance of the approach. The fact that the SDM solves nonlinear problems without using He's polynomials is a clear advantage of this technique. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach. The proposed technique has shown to computationally efficient in these examples that are important to researchers in the field of applied sciences.

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