Chebyshev-Sinc Collocation Schemes for Solving a Class of Convection Diffusion Equations

M. A. Ramadan, Talaat S. EL-Danaf, Hanem Galal

Abstract-Tthis paper, is concerned with obtaining numerical solutions for a class of convection-diffusion equations (CDEs) with variable coefficients. Our approaches are based on collocation methods. These approaches implementing all four kinds of shifted Chebyshev polynomials in combination with Sinc functions to introduce an approximate solution for CDEs . This approximate solution can be expressed as a finite double summation from the product of Sinc functions and shifted Chebyshev polynomials. The time derivatives for all four kinds of shifted Chebyshev polynomials are expressed here as linear combinations from Chebyshev polynomials themselves. New formulas for the integer derivatives with respect to time t and space x, respectively of the unknown function with two variables is expressed in terms of the product of Sinc functions and shifted Chebyshev polynomials themselves also. Special attention is given to the numerical results obtained by the proposed approaches in order to demonstrate the accuracy and efficiency of the newly proposed approaches.

Keywords: Chebyshev polynomials; Sinc functions - accuracy and efficiency- shifted Chebyshev polynomials.

I. INTRODUCTION

No one can deny that, partial differential equations have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, biology, physics and engineering ([3], [24]). The existence and uniqueness of solutions to fractional differential equations have been investigated in ([11], [19]). Consequently, considerable attention has been given to the solutions of time space CDEs. Diffusion equations are used to describe phenomena of anomalous diffusion in transport processes. Most partial differential equations do not have explicit analytical solutions, so the need to obtain approximate and numerical techniques is a problem of fundamental interest to many of authors. In the past few decades, several numerical methods to solve the partial differential equations have been studied. There are different numerical techniques based on spectral methods ([2], [4], [6], [10], [22]), finite differences methods ([13], [15], [16], [26], [27], [31]), operational methods [12] and some other approaches ([8], [9], [18], [29], [28], [30]). The authors in [21] introduced a new approach based on Sinc functions and Legendre polynomials for solving FCDEs. In this paper, several new approaches are introduced, these approaches are based on a combination of sinc functions with all four kinds of shifted Chebyshev polynomials.

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The characteristic idea behind these approaches is to use the Sinc nodes as the space collocation points and the roots of all kinds of shifted Chebyshev polynomials as the time collocation points in order to reduce. CDEs to those of solving systems of linear algebraic equations. The properties of shifted Chebyshev polynomials and Sinc functions are used to evaluate the un- known coefficients. The main advantage of these operational approaches is to convert CDEs into systems of linear algebraic equations. It not only simplifies the problem but also speeds up the computation.

The main points included in this chapter are:

1. The time derivatives of all four kinds of shifted Chebyshev polynomials are expressed in terms of shifted Chebyshev polynomials themselves also.
2. The time derivatives at Sinc collocation points and the space derivatives for the unknown function u(x, t) are deduced in terms of shifted Chebyshev polynomials.

In order to show the accuracy and efficiency of the proposed approaches, we apply the new approaches in order to obtain the numerical solution for the following time-space CDEs with variable coefficients

$$\boldsymbol{D_t^{(1)}u(x,t)} + \boldsymbol{a_1(x)D_x^{(1)}u(x,t)} + \boldsymbol{a_2(x)D_x^{(2)}u(x,t)} = \boldsymbol{g(x,t)},\ \ \boldsymbol{a} < x < b,\ \ 0 \le t \le L \qquad (1)$$

II. THE PROBLEM AND ANALYTICAL SOLUTION

The problem(1) with initial and boundary conditions as follows

with initial condition

$$u(x,0) = f(x) \qquad (2)$$

and boundary conditions

$$u(a,t) = u(b,t) \qquad (3)$$

Such that $a_{1(x)}$, $a_{2(x)}$, are continuous functions. The structure of this paper is arranged in the following way: In section 2, we introduce the properties of all four kinds of shifted Chebyshev polynomials. In section 3, we introduce the fundamental relations and the description of the new approaches. In section4, the collocation method based on the proposed approaches is introduced. In section 5, numerical test examples for CDEs equations are given al differential equations, so we try to provide numerical methods to solve such problems. to show the accuracy and efficiency of the presented approaches. Finally, In section 6, the report ends with a brief conclusion and some remarks.

III. PRELIMINARIES AND NOTATIONS

The shifted Chebyshev polynomials

The well-known shifted Chebyshev polynomials are defined on the interval [0, L]. We deal with all four kinds of shifted Chebyshev polynomials as follows:

$1^{.st}$ Kind shifted Chebyshev polynomials.

The explicit formulae of the 1.st kind shifted Chebyshev polynomials which are denoted by $T_j^*(t)$ of degree j is given by:

$$T_j^*(t) = \sum_{k=0}^{j} \frac{(-1)^{j-k}\Gamma\left(\frac{1}{2}\right) j!\,\Gamma(j+k)}{\Gamma(j)\Gamma\left(k+\frac{1}{2}\right)(j-k)!\,k!\,L^k} t^k \tag{4}$$

Where, $T_j^*(0) = (-1)^j$, $T_j^*(L) = 1$. The orthogonality condition for these polynomials is:

$$\int_0^L T_j^*(t)T_k^*(t)\omega(t)\,dt = \delta_{jk}\, h_{1k}\,, \tag{5}$$

where, the weight function $\omega(t) = \frac{1}{\sqrt{Lt-t^2}}$, $h_{1k} = \frac{a_k}{2}\pi$, $with\ a_0 = 2,\quad a_k = 1,\quad k \geq 1.$

The function *u(t)* which belongs to the space of square integrable in *[0, L]*, may be expressed in terms of shifted Chebyshev polynomials of the first kind as:

$$\boldsymbol{u(t) = \sum_{i=0}^{\infty} c_i T_i^*(t)} \tag{6}$$

where the coefficients c_i are given by:

$$\boldsymbol{c_i = \frac{1}{h_{1i}} \int_0^L u(t)T_i^*(t)\omega(t)\,dt\,,\quad i = o,1,2,\ldots}$$

$2^{.nd}$ Kind shifted Chebyshev polynomials.

The explicit formulae of the 2nd kind shifted Chebyshev polynomials which are denoted by $U_j^*(t)$ of degree j is given by:

$$U_j^*(t) = \sum_{k=0}^{j} \frac{(-1)^{j-k}\Gamma\left(\frac{1}{2}\right)(j+1)!\,\Gamma(j+k+2)}{\Gamma(j+2)\Gamma\left(k+\frac{3}{2}\right)(j-k)!\,k!\,L^k} \tag{7}$$

where, $U_j^*(0) = 2j(-1)^j$, $U_j^*(L) = 2j$. The orthogonality condition of these polynomials is:

$$\int_0^L U_j^*(t)U_k^*(t)\omega(t)\,dt = \delta_{jk}\, h_{2k}\,, \tag{8}$$

where, the weight function

$$\omega(t) = \sqrt{Lt-t^2}, h_{2k} = \frac{(k+1)!\,\Gamma\left(\frac{1}{2}\right)L^2\Gamma\left(k+\frac{3}{2}\right)}{\Gamma\left(k+\frac{3}{2}\right)(2k+2)!\,k!\,\Gamma(k+2)}.$$

The function *u(t)* which belongs to the space of square integrable in *[0, L]*, may be expressed in terms of shifted Chebyshev polynomials of the second kind as

$$u(t) = \sum_{i=0}^{\infty} c_i\, U_i^*(t),$$

where the coefficients c_i are given by:

$$c_i = \frac{1}{h_{2i}} \int_0^L u(t)U_i^*(t)\omega(t)\,dt\,,\ \ i = o,1, \tag{9}$$

$3^{.rd}$ Kind shifted Chebyshev polynomials.

The explicit formulae of the 3rd kind shifted Chebyshev polynomials which are denoted

By $V_j^*(t)$ of degree j is given by:

$$V_j^*(t) = \sum_{k=0}^{j} \frac{(2j)!!(-1)^{j-k}\Gamma\left(J+\frac{1}{2}\right)\Gamma(j+k+1)}{(2j-1)!!\Gamma(j+1)\Gamma\left(k+\frac{1}{2}\right)(j-k)!\,k!\,L^k} t^k\,. \tag{10}$$

Where

$$V_j^*(0) = \frac{(2j)!!\,(-1)^j\,\Gamma\left(J+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)(2j-1)!!\,j!},$$

$$V_j^*(L) = \frac{(2j)!!\,\Gamma\left(J+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)(2j-1)!!\,j!}.$$

The orthogonality condition of these polynomials is:

$$\int_0^L V_j^*(t)V_k^*(t)\omega(t)\,dt = \delta_{jk}\, h_{3k}\,, \tag{11}$$

where, the weight function

$$\omega(t) = x^{-0.5}(L-x)^{0.5}, h_{3k} = \frac{(2k)!!\,L\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{(2k-1)!!(2k+1)!\,k!\,\Gamma(k+1)}.$$

The function V (t) which belongs to the space of square integrable in *[0, L]*, may be expressed in terms of shifted Chebyshev polynomials of the third kind as:

$$u(t) = \sum_{i=0}^{\infty} c_i\, V_i^*(t),$$

where the coefficients c_i are given by:

$$c_i = \frac{1}{h_{3i}} \int_0^L u(t)V_i^*(t)\omega(t)\,dt\,,\qquad i = o,1,2,\ldots \tag{12}$$

4^{th} kind shifted Chebyshev polynomials.

The explicit formulae of the 4.th kind shifted Chebyshev polynomials which are denoted

By $W_j^*(t)$ of degree j is given by:

$$W_j^*(t) = \sum_{k=0}^{j} \frac{(2j)!!(-1)^{j-k}\Gamma\left(j+\frac{3}{2}\right)\Gamma(j+k+1)}{(2j-1)!!\Gamma(j+1)\Gamma\left(k+\frac{3}{2}\right)(j-k)!\,k!\,L^k} t^k\,. \tag{13}$$

where, $W_j^*(0) = \dfrac{(2j)!!\,(-1)^j\,\Gamma\left(J+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)(2j-1)!!\,j!}$,

$$W_j^*(L) = \frac{(2j)!!\,\Gamma\left(J+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)(2j-1)!!\,j!}.$$

The orthogonality condition of these polynomials is:

$$\int_0^L W_j^*(t)W_k^*(t)\omega(t)\,dt = \delta_{jk}\, h_{4k}\,, \tag{14}$$

where, the weight function

$$\omega(t) = x^{0.5}(L-x)^{-0.5},\quad h_{3k} = \frac{(2k)!!\,L\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{(2k-1)!!\,(2k+1)!\,k!\,\Gamma(k+1)}$$

The function W(t) which belongs to the space of square integrable in *[0, L]*, may be expressed in terms of shifted Chebyshev polynomials of the fourth kind as:

$$u(t) = \sum_{i=0}^{\infty} c_i\, W_i^*(t),$$

where the coefficients c_i are given by:

$$c_i = \frac{1}{h_{4i}} \int_0^L u(t)W_i^*(t)\omega(t)\,dt\,,\quad i = o,1,2,\ldots \tag{15}$$

IV. SINC FUNCTIONS PROPERTIES

In this subsection, we state the main properties of Sinc functions which will be used here. Sinc functions are used in the literature to solve numerically some kinds of differential equations ([7], [20], [23]). The Sinc functions are defined on the whole real line -∞< x <∞:

$$Sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & for\quad x\neq 0; \\ 1, & for\quad x=o. \end{cases} \tag{16}$$

The translated Sinc functions with evenly spaced nodes are given

$$S(k,h)(x) = Sinc\left(\frac{x-kh}{h}\right)\begin{cases} \frac{\sin\left(\frac{\pi}{h}(x-kh)\right)}{\frac{\pi}{h}(x-kh)}, & for\quad x\neq kh; \\ 1, & for\quad x=kh \end{cases} \tag{17}$$

such that, h > 0 and k = 0,±1,±2, ….

Sinc functions form an interpolating set of functions, which means

$$S(k,h)(jh) = \delta_{kj} = \begin{cases} 1, & for\quad k=j; \\ 0, & for\quad k\neq j. \end{cases} \tag{18}$$

The function *f(x)* which is defined on the real axis such that *h > 0*, then Whittaker cardinal expansion of *f(x)* whenever this

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series converges has the following expression

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh)Sinc\left(\frac{x-kh}{h}\right)$$

(19)

This approximation can be defined over the finite interval *(0, L)* which used here using the one-to-one conformal map

$$\emptyset(z) = \ln\left(\frac{z}{L-z}\right) \quad (20)$$

which maps the eye-shaped region

$D_E = \left\{z = x+iy: \left|arg\left(\frac{z}{L-z}\right)\right| < d \le \frac{\pi}{2}\right\}$ onto the infinite strip D_s.

Define the range of $\psi = \Phi^{-1}$ on the real line as $\Gamma = \psi(u)\ \varepsilon\ D_E : -\infty < u < \infty$:

The image which corresponds to the uniform nodes defined on the real line $\{hk\}_{-\infty}^{\infty}$ is expressed by

$$x_k = \frac{L\ exp(kh)}{1+exp(kh)}, \qquad k = 0, \pm 1, \pm 2, ... \quad (21)$$

The basis functions on (0, L) are taken to be the composite translated Sinc functions

$$S_k(x) = S(k,h)\ o\ \emptyset(x) = Sinc\left(\frac{\emptyset(x)-kh}{h}\right) \quad (22)$$

the class of functions such that the known exponential error estimates exist for Sinc interpolation is denoted by $B(D_E)$ and is defined as follows

Definition 4. Let $B(D_E)$ be the class of functions F which are analytic in D_E , satisfy

$$\int_{\psi(t+L)} |(z)dz| \to o, \quad t \to \pm\infty,$$

Where, $L = \left\{iv: |v| < d \le \frac{\pi}{2}\right\}$ and on the boundary of D_E ,denoted by (∂D_E) satisfy

$$N(F) = \int_{\partial D_E} |f(z)| < \infty.$$

Interpolation for function in $B(D_E)$ is defined in the following theorem which shows that Sinc interpolation on $B(D_E)$ converges exponentially,

Theorem 1. If $\acute{\emptyset} \in B(D_E)$, then for all $x \in \Gamma$

$$\left|f(x) - \sum_{k=-\infty}^{\infty} f(x_k)\, S_{k(x)}\right| \le \frac{N(f\emptyset)}{2\pi d \sinh\frac{\pi d}{h}} \le \frac{2N(f\emptyset)}{\pi d} exp(-\pi d|h)$$

Moreover, if

$$|f(x)| \le C\ exp(-\beta|\emptyset(x)|), \quad x \in \Gamma,$$

for some positive constants C and Β, and if the selection

$$h = \sqrt{\frac{\pi d}{\beta N}} \le \frac{2\pi d}{\ln 2},$$

then

$$\left|f(x) - \sum_{k=-N}^{N} f(x_k)\, S_{k(x)}\right| \le C_2\sqrt{2}\ exp\left(-\sqrt{\pi d \beta N}\right), \quad x \in \Gamma,$$

Where C_2 depends only f, d and β,

The derivatives of composite Sinc functions evaluated at the nodes are given as follows, [14]

$$\delta_{kj}^{(0)} = [S(k,h)\ o\ \emptyset(x)]\ |_{x=x_j} = \begin{cases}1, & for \quad k=j;\\ 0, & for \quad k\neq j.\end{cases} \quad (23)$$

$$\delta_{kj}^{(1)} = \frac{d}{d\emptyset}[S(k,h)\ o\ \emptyset(x)]\ |_{x=x_j} = \frac{1}{h}\begin{cases}1, & for \quad k=j;\\ \frac{(-1)^{j-k}}{j-k}, & for \quad k\neq j.\end{cases} \quad (24)$$

$$\delta_{kj}^{(2r)} = \frac{1}{h^{2r}}\begin{cases}\frac{(-1)^r\pi^{2r}}{2r+1}, & for\ \ k=j\\ \frac{(-1)^{j-k}}{(j-k)^{2r}}\sum_{s=0}^{r-1}\frac{(-1)^{s+1}(2r)!}{(2s+1)!}\pi^{2s}(j-k)^{2s}, & for\ k\neq j\end{cases} \quad (25)$$

$$\delta_{kj}^{(2r+1)} = \frac{1}{h^{2r+1}}\begin{cases}0, & for \quad k=j;\\ \frac{(-1)^{j-k}}{(j-k)^{2r+1}}\sum_{s=0}^{r}\frac{(-1)^s(2r+1)!}{(2s+1)!}\pi^{2s}(j-k)^{2s}, & k\neq j\end{cases} \quad (26)$$

with r= 1,2,3 ….

for more details on Sinc functions and its properties see ([14], [25]).

V. INTEGER PARTIAL DERIVATIVES FOR CHEBYSHEV-SINC COLLOCATION METHOD

In this section, we are going to explain the fundamental relations of the different approaches which are used to solve CDEs (1-3) numerically and obtain their different collocation schemes.

The first approach

It is suggested that the solution u(x; t) can be approximated in terms of the first (n + 1) shifted Chebyshev polynomials of the first kind and *2m + 1* Sinc functions as follows

$$u_{m,n}(x,t) = \sum_{i=-m}^{m}\sum_{j=o}^{n} c_{ij}\ T_j^*(t)S_i(x)$$

(27)

where, c_{ij} are the unknown Sinc-Chebyshev coefficients. The approximate solution in (3.27) satisfies the boundary conditions (3.3) since $S_i(x)$ is equivalent to zero when x tends to *a* and *b*. To express the different sorts of derivatives with fractional and integer orders for the unknown function *u(x,t)* in terms of shifted Chebyshev polynomials and Sinc functions, we introduce the following theorem.

Theorem 2.

Let *u(x,t)* be approximated by Eq.(3.27) and also suppose that υ > 0 and x_k are the Sinc collocation points then, the following relations hold:

$$D_t^{(1)}\left(u(x_k,t)\right) \cong \sum_{d=0}^{n}\sum_{j=1}^{n}\sum_{k=1}^{j} c_{kj}\ b_{1kd}\ \boldsymbol{\xi}_{1,1}T_d^*(t), \quad (28)$$

$$D_x^{(1)}\left(u(x_k,t)\right) \cong \sum_{i=-m}^{m}\sum_{j=1}^{n} c_{ij}\ \boldsymbol{\xi}_{1,2}T_j^*(t)$$

(29)

$$D_x^{(2)}\left(u(x_k,t)\right) \cong \sum_{i=-m}^{m}\sum_{j=1}^{n} c_{ij}\ \boldsymbol{\xi}_{1,3}T_j^*(t) \quad (30)$$

where

$$\boldsymbol{\xi}_{1,1} = \frac{j!\Gamma(0,5)(-1)^{j-k}\Gamma(j+k)\Gamma(k+1)}{L^k k!\Gamma\left(k+\frac{1}{2}\right)(j-k)!\Gamma(j)\Gamma(k)}, \quad \boldsymbol{\xi}_{1,2} = \acute{\emptyset}(x_k)\delta_{i,k}^{(1)}$$

$$\boldsymbol{\xi}_{1,3} = \left(\emptyset''(x_k)\delta_{i,k}^{(1)} + \left(\emptyset'(x_k)\right)^2\delta_{i,k}^{(2)}\right)$$

Proof.

$$\boldsymbol{D_t^{(1)}\left(u(x_k,t)\right) = \sum_{i=-m}^{m}\sum_{j=1}^{n} c_{ij}S_i(x_k)D_t^{(1)}\left(T_j^*(t)\right)}, \quad (31)$$

$$D_t^{(1)}(u(x_k,t)) \cong \sum_{i=-m}^{m}\sum_{j=1}^{n}\sum_{k=1}^{j} c_{ij}\frac{j!\Gamma(0{,}5)(-1)^{j-k}\Gamma(j+k)\Gamma(k+1)}{L^k k!\Gamma\left(k+\frac{1}{2}\right)(j-k)!\Gamma(j)\Gamma(k)}\delta_{i,k}^{(0)}t^{k-1} \quad (32)$$

$$D_t^{(1)}(u(x_k,t)) \cong \sum_{j=1}^{n}\sum_{k=1}^{j} c_{Kj}\frac{j!\Gamma(0{,}5)(-1)^{j-k}\Gamma(j+k)\Gamma(k+1)}{L^k k!\Gamma\left(k+\frac{1}{2}\right)(j-k)!\Gamma(j)\Gamma(k)}t^{k-1} \quad (33)$$

Now, t^{k-1} , can be expressed approximately in terms of shifted chebyshev series of the 1^{St} kind so we have:

$$t^{k-1} \cong \sum_{d=0}^{N} b_{1kd}T_d^*(t), \quad (34)$$

where, b_{1kd} is obtained from (6) with *u(t)* = t^{k-1}, then b_{1kd} can be expressed by Hypergeometric functions as follows

$$b_{1kd} = \frac{(-1)^d L^{k-1}\Gamma\left(d+\frac{1}{2}\right)\Gamma\left(k-\frac{1}{2}\right)}{\Gamma(d+1)\Gamma(k)h_{1d}} \quad 3F_2\begin{pmatrix}-d,d,k-\frac{1}{2}\\ ;1\\ \frac{1}{2},k\end{pmatrix}, \quad (35)$$

A combination of Eqs.(33), (34) and (35) leads to the desired result (28).

$$D_x^{(1)}(u(x_k,t)) \cong \sum_{i=-m}^{m}\sum_{j=1}^{n} c_{ij}\left(D_x^{(1)}S_i(x_k)\right)\left(T_j^*(t)\right), \quad (36)$$

Which takes the form

$$D_x^{(1)}(u(x_k,t)) \cong \sum_{i=-m}^{m}\sum_{j=1}^{n} c_{ij}\left(\phi'(x_k)\frac{d}{d\phi}S_i(x_k)\right)T_j^*(t). \quad (37)$$

A combination of Eqs. (36) and (37) leads to the desired result (29).

$$D_x^{(2)}(u(x_k,t)) \cong \sum_{i=-m}^{m}\sum_{j=1}^{n} c_{ij}\left(D_x^{(2)}S_i(x_k)\right)\left(T_j^*(t)\right), \quad (38)$$

$$D_x^{(2)}(u(x_k,t)) \cong \sum_{i=-m}^{m}\sum_{j=1}^{n} c_{ij}\Bigg(\phi''(x_k)\frac{d}{d\phi}S_i(x_k) +$$

∅′xk2d2d∅2SixkTj∗t. (39)

A combination of Eqs. (38) and (39) leads to the desired result (30).

The second approach

It is suggested that the solution *u(x, t)* can be approximated in terms of the first *(n + 1)* shifted Chebyshev polynomials of the second kind and *2m + 1* Sinc functions as follows

$$u(x,t) = \sum_{i=-m}^{m}\sum_{j=1}^{n} c_{ij}U_j^*(t)S_i(x), \quad (40)$$

where, c_{ij} are the unknown Sinc-Chebyshev coefficients. The approximate solution in (40) satisfies the boundary conditions (3) since $S_i(x)$ is equivalent to zero when *x* tends to *a* and *b*. To express the different sorts of derivatives with fractional and integer orders for the unknown function *u(x,t)* in terms of shifted Chebyshev polynomials of the second kind and Sinc functions, we introduce the following theorem.

Theorem 3.

Let u(x,t) be approximated by Eq.(40) and also suppose that $\upsilon > 0$ and x_k are the Sinc collocation points then, the following relations hold:

$$D_t^{(1)}(u(x_k,t)) \cong \sum_{d=0}^{n}\sum_{j=1}^{n}\sum_{k=1}^{j} c_{kj}\ b_{2kd}\,\xi_{2,1}U_d^*(t), \quad (41)$$

$$D_x^{(1)}(u(x_k,t)) \cong \sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\,\xi_{2,2}U_j^*(t), \quad (42)$$

$$D_x^{(2)}(u(x_k,t)) \cong \sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\,\xi_{2,3}U_j^*(t). \quad (43)$$

Where

$$\xi_{1,1} = \frac{j!\Gamma(0{,}5)(-1)^{j-k}\Gamma(j+k)\Gamma(k+1)}{L^k k!\Gamma\left(k+\frac{1}{2}\right)(j-k)!\Gamma(j)\Gamma(k)}, \qquad \xi_{1,2} = \acute{\phi}(x_k)\delta_{i,k}^{(1)}$$

,

$$\xi_{1,3} = \left(\phi''(x_k)\delta_{i,k}^{(1)} + \left(\phi'(x_k)\right)^2\delta_{i,k}^{(2)}\right)$$

Proof.

$$D_t^{(1)}(u(x_k,t)) = \sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}S_i(x_k)D_t^{(1)}\left(U_j^*(t)\right), \quad (44)$$

$$D_t^{(1)}(u(x_k,t)) \cong \sum_{i=-m}^{m}\sum_{j=1}^{n}\sum_{k=1}^{j} c_{ij}\frac{(j+1)!\Gamma(0{,}5)(-1)^{j-k}\Gamma(j+k+2)\Gamma(k+1)}{L^k k!\Gamma\left(k+\frac{3}{2}\right)(j-k)!\Gamma(k)}\delta_{i,k}^{(0)}t^{k-1} \quad (45)$$

$$D_t^{(1)}(u(x_k,t)) \cong \sum_{j=1}^{n}\sum_{k=1}^{j} c_{Kj}\frac{(j+1)!\Gamma(0{,}5)(-1)^{j-k}\Gamma(j+k+2)\Gamma(k+1)}{L^k k!\Gamma\left(k+\frac{3}{2}\right)(j-k)!\Gamma(k)}t^{k-1} \quad (46)$$

Now, t^{k-1} can be expressed approximately in terms of shifted chebyshev series of the $\mathbf{2}^{nd}$ kind so we have:

$$t^{k-1} \cong \sum_{d=0}^{N} b_{2kd}U_d^*(t),$$

where, b_{2kd} is obtained from (6) with u(t)= t^{k-1}, then b_{2kd} can be expressed by Hypergeometric functions as follows

$$b_{2kd} = \frac{(-1)^d L^{k+1}\Gamma\left(d+\frac{3}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(d+1)\Gamma(k+2)h_{2d}} \quad 3\,F_2\begin{pmatrix}-d,d,k+\frac{1}{2}\\ ;1\\ \frac{3}{2},k+2\end{pmatrix}, \quad (48)$$

A combination of Eqs.(46), (47) and (48) leads to the desired result (41).

$$D_x^{(1)}(u(x_k,t)) \cong \sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\left(D_x^{(1)}S_i(x_k)\right)\left(U_j^*(t)\right), \quad (49)$$

$$D_x^{(1)}(u(x_k,t)) \cong \sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\left(\phi'(x_k)\frac{d}{d\phi}S_i(x_k)\right)U_j^*(t). \quad (50)$$

A combination of Eqs. (49) and (50) leads to the desired result (42).

$$D_x^{(2)}(u(x_k,t)) \cong \sum_{i=-m}^{m}\sum_{j=1}^{n} c_{ij}\left(D_x^{(2)}S_i^{x_k}\right)\left(U_j^*(t)\right), \quad (51)$$

$$D_x^{(2)}\big(u(x_k,t)\big) \cong \sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\left(\emptyset''(x_k)\frac{d}{d\emptyset}S_i(x_k)+\right.$$

$\emptyset' xk2d2d\emptyset 2SixkUj{*}t$ (52)

A combination of Eqs. (51) and (52) leads to the desired result (43).

The third approach

It is suggested that the solution *u(x, t)* can be approximated in terms of the first *(n + 1)* shifted Chebyshev polynomials of the third kind and *2m + 1* Sinc functions as follows

$$u(x,t)=\sum_{i=-m}^{m}\sum_{j=1}^{n} c_{ij}V_j^*(t)S_i(x), \qquad (53)$$

where, c_{ij} are the unknown Sinc-Chebyshev coefficients. The approximate solution in (53) satisfies the boundary conditions (3) since $S_i(x)$ is equivalent to zero when *x* tends to *a* and *b*. To express the different sorts of derivatives with fractional and integer orders for the unknown function *u(x, t)* in terms of shifted Chebyshev polynomials of the third kind and Sinc functions, we introduce the following theorem.

Theorem 4.

Let *u(x,t)* be approximated by Eq.(40) and also suppose that $\upsilon > 0$ and x_k are the Sinc collocation points then, the following relations hold:

$$D_t^{(1)}\big(u(x_k,t)\big)\cong \sum_{d=0}^{n}\sum_{j=\lceil\upsilon\rceil}^{n}\sum_{k=1}^{j} c_{kj}\, b_{3kd}\, \boldsymbol{\xi}_{3,1}V_d^*(t), \qquad (54)$$

$$D_x^{(1)}\big(u(x_k,t)\big)\cong \sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\, \boldsymbol{\xi}_{3,2}V_j^*(t), \qquad (55)$$

$$D_x^{(2)}\big(u(x_k,t)\big)\cong \sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\, \boldsymbol{\xi}_{3,3}V_j^*(t). \qquad (56)$$

Where

$$\boldsymbol{\xi}_{3,1}=\frac{(2J)!!\,(-1)^{j-k}\Gamma\left(J+\frac{1}{2}\right)\Gamma(j+k+1)\Gamma(k+1)}{(2J-1)!!\,\Gamma\left(k+\frac{1}{2}\right)L^k k!\,(j-k)!\,\Gamma(j+1)\Gamma(k)}$$

$$\boldsymbol{\xi}_{3,2}=\acute{\emptyset}(x_k)\delta_{i,k}^{(1)}$$

$$\boldsymbol{\xi}_{3,3}=\left(\emptyset''(x_k)\delta_{i,k}^{(1)}+\big(\emptyset'(x_k)\big)^2\delta_{i,k}^{(2)}\right)$$

Proof.

$$D_t^{(1)}\big(u(x_k,t)\big)=\sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}S_i(x_k)D_t^{(1)}\left(V_j^*(t)\right), \qquad (57)$$

$$D_t^{(1)}\big(u(x_k,t)\big)\cong\sum_{i=-m}^{m}\sum_{j=1}^{n}\sum_{k=1}^{j} c_{ij}\frac{(2J)!!\,(-1)^{j-k}\Gamma\left(J+\frac{1}{2}\right)\Gamma(j+k+1)\Gamma(k+1)}{(2J-1)!!\,\Gamma\left(k+\frac{1}{2}\right)L^k k!\,(j-k)!\,\Gamma(J+1)\Gamma(k)}t^{k-1} \qquad (58)$$

$$D_t^{(1)}\big(u(x_k,t)\big)\cong\sum_{j=1}^{n}\sum_{k=1}^{j} c_{Kj}\frac{(2J)!!(-1)^{j-k}\Gamma\left(J+\frac{1}{2}\right)\Gamma(j+k+1)\Gamma(k+1)}{(2J-1)!!\Gamma\left(k+\frac{1}{2}\right)L^k k!(j-k)!\Gamma(J+1)\Gamma(k)}t^{k-1} \qquad (59)$$

Now, t^{k-1} can be expressed approximately in terms of shifted chebyshev series of the 3,rd kind so we have:

$$t^{k-1}\cong\sum_{d=0}^{N} b_{3kd}V_d^*(t), \qquad (60)$$

where, b_{3kd} is obtained from (12) with $u(t) = t^{k-1}$, then b_{3kd} can be expressed by Hypergeometric functions as follows

$$b_{3kd}=\frac{(-1)^d L^k\Gamma\left(\frac{3}{2}\right)\Gamma\left(d+\frac{1}{2}\right)\Gamma\left(k-\frac{1}{2}\right)}{\Gamma(d+1)\Gamma(k+1)\Gamma\left(\frac{1}{2}\right)h_{3d}}\;\; 3\,F_2\begin{pmatrix}-d,d,k-\frac{1}{2}\\ ;1\\ \frac{1}{2},k+1\end{pmatrix}, \qquad (61)$$

A combination of Eqs.(60), (47) and (61) leads to the desired result (54).

$$D_x^{(1)}\big(u(x_k,t)\big)\cong\sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\left(D_x^{(1)}S_i(x_k)\right)\left(V_j^*(t)\right), \qquad (62)$$

$$D_x^{(1)}\big(u(x_k,t)\big)\cong\sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\left(\emptyset'(x_k)\frac{d}{d\emptyset}S_i(x_k)\right)V_j^*(t). \qquad (63)$$

A combination of Eqs. (62) and (63) leads to the desired result (55).

$$D_x^{(2)}\big(u(x_k,t)\big)\cong\sum_{i=-m}^{m}\sum_{j=1}^{n} c_{ij}\left(D_x^{(2)}S_i^{x_k}\right)\left(V_j^*(t)\right), \qquad (64)$$

$$D_x^{(2)}\big(u(x_k,t)\big) \cong \sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\left(\emptyset''(x_k)\frac{d}{d\emptyset}S_i(x_k)+\right.$$

$\emptyset' xk2d2d\emptyset 2SixkVj{*}t$ (65)

A combination of Eqs. (64) and (65) leads to the desired result (56).

The fourth approach

It is suggested that the solution *u(x, t)* can be approximated in terms of the first *(n + 1)* shifted Chebyshev polynomials of the fourth kind and *2m + 1* Sinc functions as follows

$$u(x,t)=\sum_{i=-m}^{m}\sum_{j=1}^{n} c_{ij}W_j^*(t)S_i(x), \qquad (66)$$

where, c_{ij} are the unknown Sinc-Chebyshev coefficients. The approximate solution in (66) satisfies the boundary conditions (3) since $S_i(x)$ is equivalent to zero when x tends to a and b. To express the different sorts of derivatives with fractional and integer orders for the unknown function u(x, t) in terms of shifted Chebyshev polynomials of the fourth kind and Sinc functions, we introduce the following theorem.

Theorem 5.

Let u(x , t) be approximated by Eq.(66) and also suppose that x_k are the Sinc collocation points then, the following relations hold:

$$D_t^{(1)}\big(u(x_k,t)\big)\cong\sum_{d=0}^{n}\sum_{j=1}^{n}\sum_{k=1}^{j} c_{kj}\, b_{4kd}\,\boldsymbol{\xi}_{4,1}W_d^*(t), \qquad (67)$$

$$D_x^{(1)}\big(u(x_k,t)\big)\cong\sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\,\boldsymbol{\xi}_{4,2}W_j^*(t), \qquad (68)$$

$$D_x^{(2)}\big(u(x_k,t)\big)\cong\sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\,\boldsymbol{\xi}_{4,3}W_j^*(t). \qquad (69)$$

Where

$$\boldsymbol{\xi}_{4,1} = \frac{(2j)!!\,(-1)^{j-k}\Gamma\left(j+\frac{3}{2}\right)\Gamma(j+k+1)\Gamma(k+1)}{(2j-1)!!\,\Gamma\left(k+\frac{3}{2}\right)L^k k!\,(j-k)!\,\Gamma(j+1)\Gamma(k)}$$

$$\boldsymbol{\xi}_{4,2} = \acute{\emptyset}(x_k)\delta_{i,k}^{(1)}$$

$$\boldsymbol{\xi}_{4,3} = \left(\emptyset''(x_k)\delta_{i,k}^{(1)} + \left(\emptyset'(x_k)\right)^2\delta_{i,k}^{(2)}\right)$$

Proof.

$$D_t^{(1)}\big(u(x_k,t)\big) = \sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\,S_i(x_k)D_t^{(1)}\left(W_j^*(t)\right) \quad (70)$$

$$D_t^{(1)}\big(u(x_k,t)\big) \cong \sum_{i=-m}^{m}\sum_{j=1}^{n}\sum_{k=1}^{j} c_{ij}\,\frac{(2j)!!\,(-1)^{j-k}\Gamma\left(j+\frac{3}{2}\right)\Gamma(j+k+1)\Gamma(k+1)}{(2j-1)!!\,\Gamma\left(k+\frac{3}{2}\right)L^k k!\,(j-k)!\,\Gamma(j+1)\Gamma(k)}\,t^{k-1}$$

(71)

$$D_t^{(1)}\big(u(x_k,t)\big) \cong \sum_{j=1}^{n}\sum_{k=1}^{j} c_{Kj}\,\frac{(2J)!!(-1)^{j-k}\Gamma\left(J+\frac{3}{2}\right)\Gamma(j+k+1)\Gamma(k+1)}{(2J-1)!!\Gamma\left(k+\frac{3}{2}\right)L^k k!(j-k)!\Gamma(J+1)\Gamma(k)}\,t^{k-1} \quad (72)$$

Now, t^{k-1} can be expressed approximately in terms of shifted chebyshev series of the 4.th kind so we have:

$$t^{k-1} \cong \sum_{d=0}^{N} b_{4kd}\,W_d^*(t), \quad (73)$$

where, b_{4kd} is obtained from (15) with $u(t) = t^{k-1}$, then b_{4kd} can be expressed by Hypergeometric functions as follows

$$b_{4kd} = \frac{(-1)^d L^k \Gamma\left(\frac{1}{2}\right)\Gamma\left(d+\frac{3}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(d+1)\Gamma(k+1)\Gamma\left(\frac{3}{2}\right)h_{4d}}\;\; 3\,F_2\begin{pmatrix}-d,d,k+\frac{1}{2}\\ ;1\\ \frac{3}{2},k+1\end{pmatrix}, \quad (74)$$

A combination of Eqs.(72), (73) and (74) leads to the desired result (67).

$$D_x^{(1)}\big(u(x_k,t)\big) \cong \sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\left(D_x^{(1)}S_i(x_k)\right)\left(W_j^*(t)\right), \quad (75)$$

$$D_x^{(1)}\big(u(x_k,t)\big) \cong \sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\left(\emptyset'(x_k)\frac{d}{d\emptyset}S_i(x_k)\right)W_j^*(t). \quad (76)$$

A combination of Eqs. (75) and (76) leads to the desired result (68).

$$D_x^{(2)}\big(u(x_k,t)\big) \cong \sum_{i=-m}^{m}\sum_{j=1}^{n} c_{ij}\left(D_x^{(2)}S_i^{x_k}\right)\left(W_j^*(t)\right),$$

(77)

$$D_x^{(2)}\big(u(x_k,t)\big) \cong \sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\left(\emptyset''(x_k)\frac{d}{d\emptyset}S_i(x_k) + \left(\left(\emptyset'(x_k)\right)^2\frac{d^2}{d\emptyset^2}S_i(x_k)\right)\right)W_j^*(t)(3.78)$$

A combination of Eqs. (77) and (78) leads to the desired result (69).

VI. THE COLLOCATION SCHEME FOR EQS. (1-3) USING THE 1.st OR 2.nd OR 3.rd OR 4.th APPROACHES

The collocation scheme defined here is obtained using the Sinc collocation points x_k and the roots of shifted Chebyshev polynomials $Ch_{n+1}^*(t)$ which may be $T_{n+1}^*(t)$, $U_{n+1}^*(t)$, $V_{n+1}^*(t)$ and $W_{n+1}^*(t)$ $which$. which are denoted by

Substitute by Eq.(27) or Eq.(40) or Eq.(53) or Eq.(66) into Eq.(1) and using Eqs.(28-30) from Theorem 1 or Eqs.(41-43) from Theorem 2 or Eqs.(54-56) from Theorem 3 or Eqs.(67- 69) from Theorem 4 gives:

$$\sum_{d=0}^{n}\sum_{j=1}^{n}\sum_{k=1}^{j} c_{kj}\,b_{rkd}\,\boldsymbol{\xi}_{1,1}Ch_d^*(t_s) + a_1(x_k)\sum_{i=-m}^{m}\sum_{j=1}^{n} c_{ij}\,\boldsymbol{\xi}_{1,2}Ch_j^*(t_s) + a_2(x_k)\sum_{i=-m}^{m}\sum_{j=1}^{n} c_{ij}\,\boldsymbol{\xi}_{1,3}Ch_j^*(t_s) = g(x_k,t_s),$$

$$k = -m,\dots,m,\; s = 1,\dots,n,\; r = 1,\dots,4 \quad (79)$$

Also, substitute by Eq.(27) or Eq.(40) or Eq.(66) into Eq.(2) gives

$$\sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\,Ch_j^*(0)S_i(x) = f(x) \quad (80)$$

Collocate Eq.(80) at $2m+1$ sinec collocation points x_k gives

$$\sum_{j=0}^{n} c_{kj}\,Ch_j^*(0) = f(x_k), \quad k = -m,\dots,m, \quad (81)$$

Solving the system of $(n+1)(2m+1)$ linear algebraic equations Eqs.(97-81) gives the unknown coefficients c_{ij} directly, the approximate numerical solution $u_{m,n}(x,t)$ gives in Eq.(27) can be calculated.

VII. NUMERICAL TEST EXAMPLES

In order to illustrate the effectiveness of the proposed method , we implement it to solve the following test examples. In all examples, we choose $\beta = 1$ and $d = \frac{\pi}{2}$ $then$ $h = \frac{\pi}{\sqrt{2N}}$.

Example 1 :

Consider the following time-partial diffusion equation

$$D_t^{(1)}u(x,t) - D_x^{(2)}u(x,t) = g(x,t), \quad 0 < x < 1, \quad 0 \le t \le 1, \quad (82)$$

Where ,

$$g(x,t) = \frac{2}{\Gamma(3-\upsilon)}t^{2-\upsilon}\sin(2\pi x) + 4\pi^2t^2\sin(2\pi x).$$

with initial condition

$$u(x,0)=0, \quad (83)$$

and boundary conditions

$$u(0,t) = u(1,t)=0. \quad (84)$$

The exact solution for this problem is

$$u(x,t) = t^2\sin(2\pi x).$$

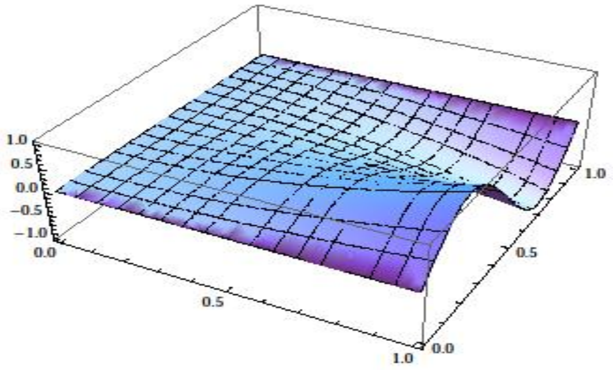

Figure 1. Plot of the 3D solution *u(x, t) at m = 6, n = 3 using the first approach*

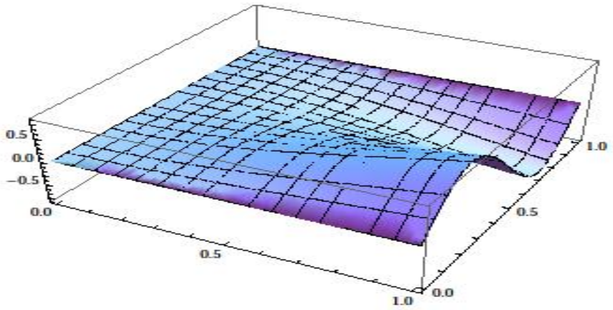

Figure 2. Plot of the 3D solution u(x, t) at m = 6, n = 3 using the second approach

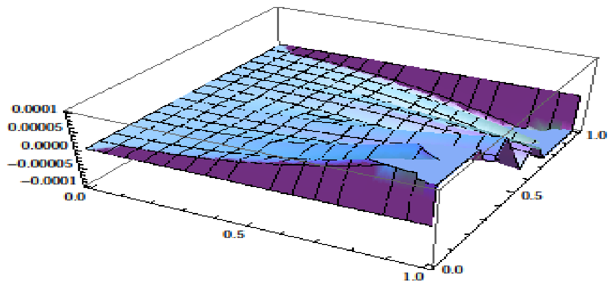

Figure 3. Plot of the 3D solution u(x, t) at m = 6, n = 3 using the third approach

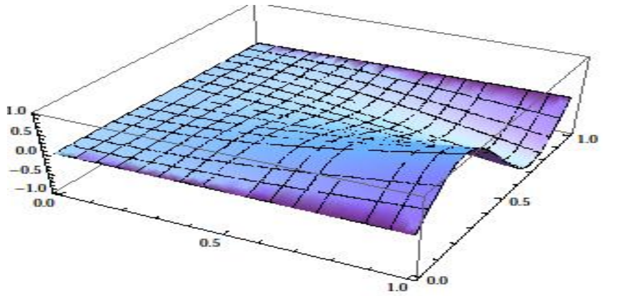

Figure 4. Plot of the absolute error at m = 6, n = 3 using the fourth approach

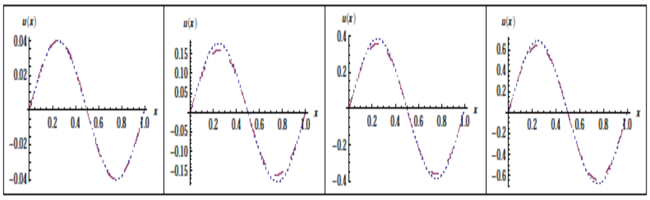

Figure 5. Plot of the truncated solution u(x) with m = 6; n = 3 at t = 0:2; 0:4; 0:6; 0:8 using the 1.st, 2.nd, 3.rd and 4.th approaches from left to right, respectively

Example 2:

Consider the following initial boundary value problem of partial differential equation

$$D_t^{(1)}u(x,t) + D_x^{(1)}u(x,t) + D_x^{(2)}u(x,t) = 2t + 2x^2 + 2,\ 0 < x < 1,\ 0 \le t \le 1, 0 < \upsilon \le 1 \quad (85)$$

With initial condition

$$u(x,0) = x^2 \quad (86)$$

and boundary conditions

$$u(0,t) = 2\frac{\Gamma(2)}{\Gamma(3)}t^2, \quad u(1,t) = 1 + 2\frac{\Gamma(2)}{\Gamma(3)}t^2 \quad (87)$$

The exact solution for this problem is

$$u(x,t) = x^2 + 2\frac{\Gamma(2)}{\Gamma(3)}\, t^2$$

Reformulate this problem by using the following transformation:

$$v(x,t) = u(x,t) - x^2 + 2\frac{\Gamma(2)}{\Gamma(3)}\, t^2$$

Gives

$$D_t^{(1)}v(x,t) + xD_x^{(1)}u(x,t) + D_x^{(2)}v(x,t) = 2x^2 - x + 2 \quad (88)$$

With initial condition

$$v(x,0) = x^2 - x \quad (89)$$

and boundary conditions

$$v(0,t) = v(1,t) = 0 \quad (90)$$

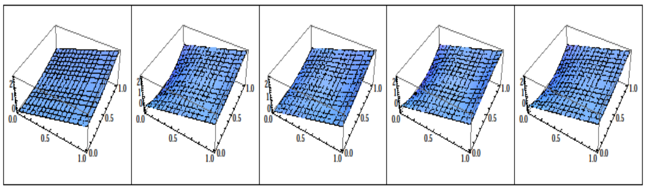

Figure 6. Plot of the exact solution and the truncated solution u(x, t) with m = 30, n = 3 using the 1.st, 2.nd, 3.rd and 4.th approaches from left to right, respectively

VIII. CONCLUTION

In this paper, we develop accurate and efficiency approaches based on Sinc functions and all four kinds of shifted Chebyshev polynomials. These approaches are used to transform every CDEs equation with variable coefficients to a system of linear algebraic equations. From the numerical test examples, it is clear that the numerical solutions obtained from these approaches are in excellent agreement with the exact solutions. For the future work, the application of these approaches will be given to solve numerically initial-boundary nonlinear partial differential equations.

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