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Super $(a,1)$-Tree-Antimagicness of Sun Graphs

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***Abstract:** Let $G=(V,E)$ be a finite simple graph with $|V(G)|$ vertices and $|E(G)|$ edges. An edge-covering of G is a family of subgraphs $H_1,H_2,\ldots,H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i, $i=1,2,\ldots,t$. If every subgraph H_i is isomorphic to a given graph H, then the graph G admits an H-covering. A graph G admitting H covering is called an (a,d)-H-antimagic if there is a bijection $f:V\cup E\rightarrow\{1,2,\ldots,|V(G)|+|E(G)|\}$ such that for each subgraph H' of G isomorphic to H, the sum of labels of all the edges and vertices belonged to H' constitutes an arithmetic progression with the initial term a and the common difference d. For $f(V)=\{1,2,3,\ldots,|V(G)|\}$, the graph G is said to be super (a,d)-H-antimagic and for $d=0$ it is called H-supermagic.*

In this paper, we investigate the existence of super $(a,1)$-S_3-antimagic labeling of Sun graphs SG_n, its uniform subdivision $SG_n(r)$, disjoint union of sun graphs and its uniform subdivision denoted by mSG_n and $mSG_n(r)$ respectively, where r, m ≥ 1.

***Keywords:** Sun graph SG_n, uniform subdivided Sun graph $SG_n(r)$, su- per (a, 1)-S3-antimagic, super (a, 1)-S3(r)-antimagic, disjoint union of Sun graph mSG_n and its uniform subdivision $mSG_n(r)$. MR (2010) Subject Classification: 05C78, 05C70*

I. INTRODUCTION

An *edge-covering* of finite and simple graph G is a family of subgraphs $H_1,H_2,\ldots,H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i, $i=1,2,\ldots,t$. In this case we say that G admits an $(H_1,H_2,\ldots,H_t)$-*(edge) covering*. If every subgraph H_i is isomorphic to a given graph H, then the graph G admits an H-*covering*. A graph G admitting an H-covering is called (a,d)-H-*antimagic* if there exists a total labeling

$$f:V(G)\cup E(G)\rightarrow\{1,2,\ldots,|V(G)|+|E(G)|\}$$

such that, for all subgraphs H' of G isomorphic to H, the H-weights,

$$wt_f(H')=\sum_{v\in V(H')}f(v)+\sum_{e\in E(H')}f(e),$$

constitute an arithmetic progression $a,a+d,a+2d,\ldots,a+(t-1)d$, where $a>0$ and $d\geq 0$ are two integers, and t is the number of all subgraphs of G isomorphic to H. Moreover, G is said to be *super* (a,d)-H-*antimagic*, if the smallest possible labels appear on the vertices. If G is a (super) (a,d)-H-antimagic graph then the corresponding total labeling f is called the *(super)* (a,d)-H-*antimagic labeling*. For $d=0$, the (super) (a,d)-H-antimagic graph is called H-*magic* and H-*supermagic*, respectively.

The (super) H-magic graph was first introduced by Gutiérrez and Lladó in [9]. They proved that the star $K_{1,n}$ and the complete bipartite graphs $K_{n,m}$ are $K_{1,h}$-supermagic for some h. They also proved that the path P_n and the cycle C_n are P_h-supermagic for some h. Lladó and Moragas [15] investigated C_n-(super)magic graphs and proved that wheels, windmills, books and prisms are C_h-magic for some h. Some results on C_n-supermagic labelings of several classes of graphs can be found in [19]. Maryati et al. [17] gave P_h-(super)magic labelings of some trees such as shrubs, subdivision of shrubs and banana tree graphs. Other examples of H- supermagic graphs with different choices of H have been given by Jeyanthi and Selvagopal in [11]. Maryati et al. [16] investigated the G-supermagicness of a disjoint union of c copies of a graph G and showed that disjoint union of any paths is cP_h-supermagic for some c and h.

The (a,d)-H-antimagic labeling was introduced by Inayah et al. [9]. In [10] Inayah et al. investigated the super (a,d)-H- antimagic labelings for some shackles of a connected graph H.

For $H\cong K_2$, (super) (a,d)-H- antimagic labelings are also called (super) (a,d)- edge - antimagic total labelings. For further information on (super) edge-magic labelings, one can see [4, 5, 7, 14]. The (super) (a,d)-H-antimagic labeling is related to a (super) d-antimagic labeling of type $(1,1,0)$

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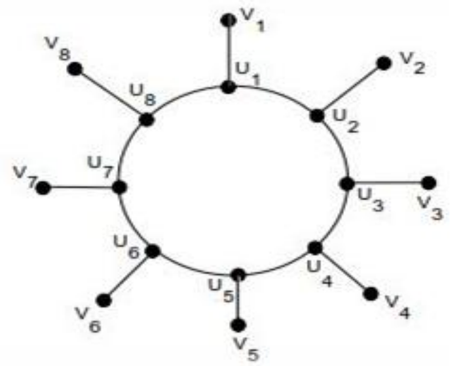
of a plane graph that is the generalization of a face-magic labeling introduced by Lih [12]. Further information on super d - antimagic labelings can be found in [2, 3, 6].

In this paper, we investigate the existence of super (a, 1)-S_3-antimagic labeling of Sun graphs SG_n, its uniform subdivision $SG_n(r)$, disjoint union of sun graphs and its uniform subdivision denoted by mSG_n and $mSG_n(r)$ respectively, where r, m ≥ 1.

II. SUPER TREE-ANTIMAGIC LABELING OF SUN GRAPHS

A star S_m is a tree consisting of one vertex adjacent to m vertices. In other words, a complete bipartite graph $K_{1,m}$ is called a Star S_m.

The Sun graph SG_n, $n \geq 3$, is obtained from a cycle C_n by attaching one (1) pendant edge at each vertex of the C_n. The vertices on the cycle will be called the *cycle vertices* and the edges of the cycle will be called the *cycle edges*. Remaining vertices and edges will be called the pendant vertices and the pendant edges.



The Sun Graph SG_n contains $2n$ vertices and edges. The vertex set $V(SG_n)$ consists of the elements $\{u_j, v_j\}$ and the edge set $E(SG_n)$ consists of the elements $\{u_j u_{j+1}\} \cup \{u_j v_j\}$, where indices are taken modulo n.

2.1. Super $(a,1)$-S_3-antimagic Labeling of Sun Graph SG_n

Let S_3 be a star on 4 vertices. Every star S_3 in SG_n has the vertex set $V(S_3^{(j)}) = \{u_{j-1}, u_j, u_{j+1}, v_j\}$ and the edge set $E(S_3^{(j)}) = \{u_{j-1}u_j, u_j u_{j+1}, u_j v_j\}$,

where indices are taken modulo n.

Under a total labeling g, for the S_3 weight of $S_3^{(j)}$, $j = 1, \ldots, n$, we have

$$wt_g(S_3^{(j)}) = \sum_{v \in V(S_3^{(j)})} g(v) + \sum_{e \in E(S_3^{(j)})} g(e).$$

$$4.3cm = \sum_{s=j-1}^{j+1} g(u_s) + \sum_{s=j-1}^{j} g(u_s u_{s+1}) + g(v_j) + g(u_j v_j) \quad (1)$$

Theorem 1 *Let $n \geq 3$ be positive integer and S_3 be a star on 4 vertices. Then Sun graph SG_n admits a super $(a,1)$-S_3-antimagic labeling.*

Proof. Consider the total labeling g_1 defined as:

$$g_1(u_j) = j \qquad j = 1,2,\ldots,n$$
$$g_1(v_j) = 2n + 1 - j \qquad j = 1,2,\ldots,n$$
$$g_1(u_j u_{j+1}) = 3n + 1 - j \qquad j = 1,2,\ldots,n-1$$
$$g_1(u_j v_j) = 3n + j \qquad j = 1,2,\ldots,n$$

Above assignment of numbers shows g_1 is a super labeling since vertices are labeled with integers $\{1,2,\ldots,2n\}$.

Under the labeling g_1 the pendent edges receive labels $\{3n+1,\ldots,4n\}$ and the cycle edges receive labels $\{2n+1,\ldots,3n\}$.

Moreover, sum of the pendent edge label and its end vertex (degree 1) label is constant. More precisely, we have

$$g_1(v_k) + g_1(u_k v_k) = 5n + 1 \quad (2)$$

Also, the cycle C_n is P_3 antimagic, that is,

$$\sum_{k=j-1}^{j+1} g(u_k) + \sum_{k=j-1}^{j} g(u_k u_{k+1}) = 6n + 3 + j \quad (3)$$

where indices are taken modulo n.

According to (1), (2) and (3), we obtain

$$wt_{g_1}(S_3^{(j)}) = 11n + 4 + j$$

Thus under the labeling g_1

$$\{wt_{g_1}(S_3^{(j)}) : 1 \leq j \leq n\} = \{A + j : 1 \leq j \leq n\}$$

where $A = 11n + 4 + j$.

This concludes the proof.

2.2. Super $(a,1)$-$S_3(r)$ antimagic Labeling of unifromly Subdivided Sun Graphs $SG_n(r)$

Let $SG_n(r)$ be a graph obtained from SG_n such that into every edge of SG_n are inserted $r \geq 1$ new vertices.

The graph $SG_n(r)$ contains $2n(r+1)$ vertices and edges. The vertex set $V(SG_n(r))$ consists of the elements $\cup_{j=1}^{n} \left\{\{u_j, v_j\} \cup \{u_{j,s}, v_{j,s} : 1 \leq s \leq r\}\right\}$

and the edge set $E(SG_n(r))$ consists of the elements

$\{u_j u_{j,1}\} \cup \{u_{j,r} u_{j+1}\} \cup \{v_j v_{j,1}\} \cup \{v_{j,r} u_j\} \bigcup_{s=1}^{r-1} \{u_{j,s} u_{j,s+1}, v_{j,s} v_{j,s+1}\},$

where indices are taken modulo n.

Let $S_3(r)$ be a graph on $3(r+1)+1$ vertices obtained from the star graph S_3 such that into every edge of S_3 are inserted $r \ge 1$ new vertices. Every graph $S_3(r)$ in $SG_n(r)$ has the vertex set

$V(S_3^{(j)}(r)) = \cup_{k=j-1}^{j+1}\{u_k\} \cup_{k=j-1}^{j} \cup_{s=1}^{r}\{u_{k,s}\} \cup_{s=1}^{r}\{v_j, v_{j,s}\}$

And the edge set

$$E(S_3^{(j)}(r)) = \cup_{s=1}^{r-1}\left\{\cup_{k=j-1}^{j}\{u_k u_{k,1}, u_{k,s} u_{k,s+1}\}, \{v_{j,s} v_{j,s+1}\}\right\}$$

$$\cup_{k=j-1}^{j}\{u_{k,r} u_k + 1\} \cup \{v_j v_{j,1}, v_{j,r} u_j\}$$

where indices are taken modulo n.

For the $S_3(r)$ weight of $S_3^{(j)}(r)$ under a total labeling h we have

$$wt_h(S_3^{(j)}(r)) = \sum_{v \in V(S_{3(r+1)}^{(j)})} h(v) + \sum_{e \in E(S_{3(r+1)}^{(j)})} h(e)$$

$$= \sum_{k=j-1}^{j+1} h(u_k) + h(v_j) + \sum_{k=j-1}^{j}\sum_{s=1}^{r} h(u_{k,s}) + \sum_{s=1}^{r} h(v_{j,s})$$

$$+ \sum_{k=j-1}^{j} h(u_k u_{k,1}) + \sum_{k=j-1}^{j}\sum_{s=1}^{r-1} h(u_{k,s} u_{k,s+1}) + \sum_{k=j-1}^{j} h(u_{k,r} u_{k+1})$$

$$+ h(v_j v_{j,1}) + h(v_{j,r} u_j) + \sum_{s=1}^{r-1} h(v_{j,s} v_{j,s+1})$$

Theorem 2. *Let $n \ge 3$, $r \ge 1$ be positive integers and $S_3(r)$ be a graph on $3(r+1)+1$ vertices obtained from star S_3 on 4 vertices. Then uniformly subdivided Sun graph $SG_n(r)$ admits a super $(a,1)$ - $S_3(r)$ -antimagic labeling.* ***Proof.*** Consider the total labeling h define in the following way

$h(u_j) = j, \quad j = 1,2,\dots,n$

$h(v_j) = (r+2)n + 1 - j, \quad j = 1,2,\dots,n$

$h(u_j u_{j,1}) = (3r+4)n + 1 - j, \quad j = 1,2,\dots,n$

$h(u_{j,r} u_{j+1}) = 4(r+1)n + 1 - j, \quad j = 1,2,\dots$

$h(v_j v_{j,1}) = 2(r+1)n + j, \quad j = 1,2,\dots,n$

$h(v_{j,r} u_j) = (3r+2)n + j, \quad j = 1,2,\dots,n$

$h(u_{j,s}) = sn + j, \quad j = 1, 2, \dots, n$

$s = 1, 2, \dots, r,\ h(v_{j,s}) = (2r - s + 3)n + 1 - j$

$j = 1, 2, \dots, n$

$s = 1,2,\dots,r,$

$h(u_{j,s} u_{j,s+1}) = [4(r+1) - s]n + 1 - j \quad j = 1,2,\dots,n$

$s = 1,2,\dots,r-1,$

$h(v_{j,s} v_{j,s+1}) = (2r + s + 2)n + j \quad j = 1,2,\dots,n$

$s = 1, 2, \dots, r-1$

where indices are taken modulo n.

Above assignment of numbers shows h is a super labeling as vertices are labeled with integers $\{1,2,\dots,2(r+1)n\}$.

The sum of the vertices and edges labels on pendent edges is constant. More precisely, we have

$$h(v_j) + h(v_j v_{j,1}) + h(v_{j,r} u_j) +$$

$$\sum_{s=1}^{r} h(v_{j,s}) + \sum_{s=1}^{r-1} h(v_{j,s} v_{j,s+1}) = [6(r+1)n + 1 + j] + r[\frac{(3r+5)n}{2} + 1 - j] +$$

$$+(r-1)[\frac{(5r+4)n}{2} + j] \quad = 4n(r^2 + 2r + 1) + r + 1 \tag{5}$$

Under the labeling h, the cycle $C_n(r)$ is P_{2r+3} antimagic, that is,

$$\sum_{k=j-1}^{j+1} h(u_k) + \sum_{k=j-1}^{j}\sum_{s=1}^{r} h(u_{k,s}) + \sum_{k=j-1}^{j} h(u_k u_{k,1}) +$$

$$+ \sum_{k=j-1}^{j} h(u_{k,r} u_{k+1}) + \sum_{k=j-1}^{j}\sum_{s=1}^{r-1} h(u_{k,s} u_{k,s+1}) +$$

$$+ \sum_{k=1}^{r} h(v_{j,k}) + \sum_{s=1}^{r-1} h(v_{j,s} v_{j,s+1}) = [j(2r+3) + rn(r+1) - r] + \tag{4}$$

$$+ rn(7r+15) + (r+1)(3-2j) + 8n$$

$$= 8n(r^2 + 2r + 1) + (2r+3) + j$$

According to (4), (5) and (6), we obtain

$$wt_g(S_3(r)) = 12rn(r+2) + 3(4n+r) + 4 + j$$

Thus under the labeling h the set of all $S_3(r)$ -weights consists of consecutive integers. This concludes the proof.

2.3. $S_3(r)$ antimagic Labeling of disjoint union of Graphs $SG_n(r)$

In [1], the following result is proved on tree-antimagicness of disconnected graphs.

Corollary 1. *Let G be an $(a,1)$-T-antimagic graph, where T is a tree. Then the disjoint union of arbitrary number of copies of G, that is, mG, $m \geq 1$, also admits a super $(b,1)$-T-antimagic total labeling.*

In the light of corollary (1) and from theorem (1) and (2), we immediately obtain:

Corollary 2. *Let $m \geq 1$, and $n \geq 3$ be positive integers. Then m copy of SG_n, that is mSG_n, is a super $(b,1)$-S_3-antimagic.*

Corollary 3 *Let $r, m \geq 1$, and $n \geq 3$ be positive integers. Then m copy of $SG_n^{(1)}(r)$, that is $mSG_n^{(1)}(r)$, is a super $(b,1)$-$S_3(r)$-antimagic.*

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