# Integrated Inventory Models for Decaying Items with Exponential Demand under Inflation

**Kapil Kumar Bansal, Navin Ahalawat**

***Abstract: In this article we take consideration of a two warehouse inventory problems with exponential and time dependent increasing trend in demand for deteriorating items under inflation. Shortage is allowed and partially backlogged .The scheduling period is taken to be variable and not constant. The solution procedure provided here helps the decision maker to decide whether to rent a warehouse or not .The result have been validated with help of some numerical example and comprehensive sensitivity analysis has also be performed.***

***Keywords: inventory, warehouse, deterioration, partial backlogging, lost sales, inflation and variable holding cost.***

## I. INTRODUCTION

In many real-life situations, the practical experiences reveal that some but not all customers will wait for backlogged items during a shortage period, such as for fashionable commodities or high-tech products with short product life cycle. The longer the waiting time is, the smaller the backlogging rate would be. According to such phenomenon, taking the backlogging rate into account is necessary. However, most of the inventory models unrealistically assume that during stockout either all demand is backlogged or all is lost. In reality often some customers are willing to wait until replenishment, especially if the wait will be short, while others are more impatient and go elsewhere. The backlogging rate depends on the time to replenishment-the longer customers must wait, the greater the fraction of lost sales. **Abad (1996)** developed a pricing and lot-sizing EOQ model for a product with a variable rate of deterioration and partial backlogging. **Abad (2000)** then extended the optimal pricing and lot-sizing EOQ model to an economic production quantity (i.e. EPQ) model. **Papachristos** and **Skouri (2000)** discussed an optimal replenishment policy for deteriorating items with time-varying demand and partial-exponential-type backlogging. Other articles related to this research were written by **Abad (2001)**, **Ouyang** *et al.* (**2005), Jolai** *et al.* **(2006)**, **Yang (2007)** and so on.

It is necessary to consider the effects of inflation on the inventory system, as many countries experience high annual inflation rate. Besides this, inflation also influences demand of certain products. The fundamental result in the development of EOQ model with inflation is that of **Buzacott (1975)** who discussed EOQ model with inflation subject to different types of pricing policies. **Bose** *et al.* **(1995)** presented a paper on deteriorating items with linear time dependent rate and shortages under inflation and time discounting.

**Manuscript Received on July 04, 2012.**

**Dr. Kapil Kumar Bansal**, Head Research & Publication, SRM University, NCR Campus, Ghaziabad (U.P.), India

**Mr. Navin Ahalawat**, Research Scholar, JJT University Rajasthan, India

**Wee** and Law (1999) addressed the problem with finite replenishment rate of deteriorating items taking account of time value of money. Chang (2004) proposed an inventory model for deteriorating items under inflation under a situation in which the supplier provides the purchaser a permissible delay of payments if the purchaser orders a large quantity. Jaggi et al. (2006) presented the optimal inventory replenishment policy of deteriorating items under inflationary conditions. The demand rate was assumed to be a function of inflation; shortages were allowed and completely backlogged.

In general, production order quantity models are used for cases where the firm receives its inventory over a period of time or where the item is produced locally rather than purchased. These models are useful when units are continuously added to the inventory over time while the production is in process. In that case, the main decision for inventory management is to determine how much to manufacture so that the total cost is minimized. The objective of this study is to find the optimal inventory policies when inflation induced demand is considered. Demand is difficult to predict and could lead to stock-out situations and somewhere a reason of insufficient production which is provide the appropriate material to all manufacturers. In several industries, such as the production of paper, furniture, textile and food, the use of material family is a regular practice but because of the demand uncertainty, production policy could be a combination of demand and on-hand inventory which gives flexibility to production decisions allowing a set of possible formulations for the same final product.

Misra (1975) considered an EPQ model for deteriorating items with both a varying and a constant rate of deterioration. Balkhi and Benkherouf (1996) proposed a method for obtaining an optimal production cycle time of deteriorating items in a model where demand and production rates are functions of time. Goyal and Giri (2003) developed the production inventory model for perishable goods with time varying demand and production. In most of the inventory models, it is assumed that the deterioration occurs as soon as the retailer receives the commodity but for many items this is not true. Li et al. (2010) provided a comprehensive introduction about the deteriorating items inventory management research status, this paper reviews the recent studies in relevant fields.

In this article, a two-warehouse inventory problem with a exponential trend in demand and variable holding cost for deteriorating items under inflation is considered. Shortages are allowed and are partially backlogged.

The scheduling period is taken to be variable and not constant. The solution procedure provided here helps the decision-maker to decide whether to rent a warehouse or not. The results have been validated with the help of some numerical examples and comprehensive sensitivity analysis has also been performed.

2. Assumptions And Notations : The mathematical models of the two-warehouse inventory problems are based on the followed assumptions :

1. Replenishment rate is infinite.
2. Lead time is zero.
3. The OW has a fixed capacity of W units and the RW has unlimited capacity.
4. The goods of the RW are consumed only after consuming the goods kept in the OW.
5. The inventory costs (including holding cost and deterioration cost) in the RW are higher than those in the OW.
6. The demand rate $D(t)$, is an increasing and exponential function of time t, given by

   $D(t) = ae^{bt}, \quad I(t) > 0, \quad D = a, \quad I(t) = 0,$

   where *a* and *b* are demand positive constants.
7. Unsatisfied demand/shortages are allowed. Unsatisfied demand is partially backlogged, and the fraction of shortages backordered is a differentiable and decreasing function of time *t*, denoted by $\delta(t)$, where *t* is the waiting time up to the next replenishment. To take care of this situation we have defined the partial back-logging rate to be $\delta(t) = e^{-\delta t}$, when inventory is negative. The backlogging parameter $\delta$ is a positive constant.
8. With the viewpoint of cost-minimization, the opportunity cost due to lost sale is the sum of the revenue loss and the cost of goodwill. Hence, the opportunity cost due to lost sale here is greater than the unit purchase cost.
9. We are considering holding cost as time dependent

**Notations :**

| | | |
|---|---|---|
| $D(t)$ | : | Demand rate which is an exponential function of time $t(ae^{bt};\ a, b > 0)$ |
| $Q$ | : | The replenishment quantity per replenishment |
| $W$ | : | The capacity of the owned warehouse (OW) |
| $Z$ | : | The initial inventory for the period |
| $\alpha$ | : | The deterioration rate in the OW, where $0 < \alpha < 1$ |
| $\beta$ | : | The deterioration rate in the RW, where $0 < \beta < 1$ |
| $r$ | : | Discount rate, representing the time value of money |
| $i$ | : | Inflation rate |
| $R$ | : | *r*-*i*, representing the net discount rate of inflation is constant |
| $A$ | : | Replenishment cost per order for a single-warehouse system |
| $A1$ | : | Replenishment cost per order for a two warehouse system |
| $c$ | : | Purchasing cost per unit |
| $s$ | : | The shortage cost per unit time |
| $c_1$ | : | The unit opportunity cost due to lost sale, if the shortage is lost |
| $H$ | : | The holding cost per unit per unit time in the OW |
| $F$ | : | The holding cost per unit per unit time in the RW, $F > H$ |
| $TC2$ | : | The present value of the total relevant cost per unit time in a two-warehouse system |
| $TC1$ | : | The present value of the total relevant cost per unit time in a single-warehouse system |
| $I_o(t)$ | : | The inventory level in the OW at time *t* |
| $I_r(t)$ | : | The inventory level in the RW at time *t* |

| | | |
|---|---|---|
| $\delta(t)(=e^{-\delta t})$ | : | The backlogging rate, where $\delta$, the backlogging parameter is a positive constant and $t$ is the waiting time up to the next replenishment |
| $B(t)$ | : | The backlogged level at time $t$ |
| $L(t)$ | : | The number of lost sales at time $t$ |
| $t_r$ | : | The time at which the inventory level reaches zero in the RW in a two-warehouse system. (in a single-warehouse system it is the time at which the inventory level reaches zero in the OW) |
| $t_o$ | : | The time at which the inventory level reaches zero in the OW in the two-warehouse system |
| $t_s$ | : | The time at which the shortage level reaches the lowest point in the replenishment cycle in the two-warehouse system. (In a single-warehouse system also it is the time at which the shortage level reaches the lowest point). |
| $\gamma$ | : | holding cost parameter use for RW |
| $\gamma'$ | : | holding cost parameter use for OW |

**3. Two-storage Model:**

At time $t=0$, a lot size of $Q$ units enters the system from which a portion is used to meet the partial backlogged items towards previous shortages, and the initial inventory for the period is $Z$. Out of these $Z$ units, $W$ units are kept in the OW and the rest $(Z-W)$ units are stored in the RW provided $Z>W$, otherwise zero units are stored in the RW. The goods of the OW are consumed only after consuming the goods kept in the RW. By time $t_r$, inventory level in the RW reaches to zero due to the combined effect of demand and deterioration, and the inventory level $W$ in the OW also reduces due to the effect of deterioration only. During $(t_r,\ t_o)$, inventory level in the OW reaches to zero due to the combined effect of demand and deterioration. By the time $t_o$, both warehouses are empty and thereafter shortages are allowed to occur. The partially backlogged quantity is supplied to the customers at the beginning of the next cycle. By the time $t_s$, the replenishment cycle starts.

Hence, during the interval $(0,\ t_r)$, the inventory levels at time $t$ in the RW and the OW are governed by the following differential equations:

$$\frac{dI_r(t)}{dt}+\beta I_r(t)=-f(t) \qquad 0\le t\le t_r \quad \text{Where } f(t)=ae^{bt} \qquad \text{.... (1)}$$

With the boundary condition $I_r(t_r)=0$ and

$$\frac{dI_o(t)}{dt}+\alpha I_o(t)=0 \qquad 0\le t\le t_r \qquad \text{.... (2)}$$

With the initial condition $I_o(0)=W$, respectively. While during the interval $(t_r,\ t_o)$, the inventory level at time $t$ at the OW $I_o(t)$ is governed by the following differential equation:

$$\frac{dI_o(t)}{dt}+\alpha I_o(t)=-f(t) \qquad t_r\le t\le t_o \qquad \text{.... (3)}$$

With the boundary condition $I_o(t_o)=0$. Similarly during $(t_o,\ t_s)$ the backlogged level at time $t$, $B(t)$ is governed by the following differential equation.

$$\frac{dB(t)}{dt}=e^{-\delta(t_s-t)}f(t) \qquad t_o\le t\le t_s \qquad \text{.... (4)}$$

With the boundary condition $B(t_o)=0$

The solution to equations (1)-(4) are

$$I_r(t) = \frac{-ae^{bt}}{b+\beta} + \frac{ae^{(b+\beta)t_r}e^{-\beta t}}{b+\beta} \qquad 0 \le t \le t_r \qquad ....(5)$$

$$I_o(t) = We^{-\alpha t} \qquad 0 \le t \le t_r \qquad ....(6)$$

$$I_o(t) = \frac{-ae^{bt}}{b+\alpha} + \frac{ae^{(b+\alpha)t_o}e^{-\alpha t}}{b+\alpha} \qquad t_r \le t \le t_o \qquad ... (7)$$

And $$B(t) = \frac{a}{b+\delta}e^{-\delta t_s}\left[e^{(b+\delta)t} - e^{(b+\delta)t_o}\right] \quad t_o \le t \le t_s \qquad .... (8)$$

The number of lost sales at time $t$ is

$$L(t) = \int_{t_o}^{t}\left[1-e^{-\delta(t_s-t)}\right]f(t)dt \qquad t_o \le t \le t_s$$

$$L(t) = \int_{t_o}^{t}\left[ae^{bt} - ae^{(b+\delta)t}e^{-\delta t_s}\right]dt = \left[\frac{ae^{bt}}{b}\right]_{t_o}^{t} - \left[\frac{ae^{-\delta t_s}e^{(b+\delta)t}}{b+\delta}\right]_{t_o}^{t}$$

$$L(t) = \frac{a}{b}[e^{bt} - e^{bt_o}] - \frac{a}{b+\delta}e^{-\delta t_s}\left[e^{(b+\delta)t} - e^{(b+\delta)t_o}\right] \qquad .... (9)$$

Using the condition $I_r(t) = Z - W$ at $t = 0$ in equation (5) and $B(t) = Q - Z$ at $t = t_s$, we have

$$Z = W - \frac{a}{b+\beta} + \frac{ae^{(b+\beta)t_r}}{b+\beta} \qquad .... (10)$$

And $$Q = Z + \frac{a}{b+\delta}e^{-\delta t_s}\left[e^{(b+\delta)t_s} - e^{(b+\delta)t_o}\right] \qquad .... (11)$$

Using the continuity of $I_o(t)$ at $t = t_r$, we know from equation (6) and (7) that

$$I_o(t_r) = We^{-\alpha t_r} = \frac{-ae^{bt_r}}{b+\alpha} + \frac{ae^{(b+\alpha)t_o}}{b+\alpha}e^{-\alpha t_r} \qquad .... (12)$$

Which implies that $$W = \frac{a}{b+\alpha}\left[e^{(b+\alpha)t_o} - e^{(b+\alpha)t_r}\right] \qquad .... (13)$$

Since the deterioration rate $\alpha$ lies between 0 and 1. On solving

$$W + \frac{a}{b+\alpha}e^{(b+\alpha)t_r} = \frac{a}{b+\alpha}e^{(b+\alpha)t_o} \Rightarrow \frac{(b+\alpha)}{a}W + e^{(b+\alpha)t_r} = e^{(b+\alpha)t_o}$$

$$(b+\alpha)t_o = \log\left[\frac{(b+\alpha)}{a}W + e^{(b+\alpha)t_r}\right],\ t_o = \frac{1}{b+\alpha}\log\left[\frac{(b+\alpha)}{a}W + e^{(b+\alpha)t_r}\right] \qquad ....(14)$$

We note that $t_o$ is a function of $t_r$, therefore $t_o$ is not a decision variable.

**Now the total cost consists the following components:**

Inventory holding cost in the RW.

Inventory holding cost in the OW.

Replenishment cost

Backlogging cost

Lost sales cost

Deterioration cost of items in the RW and the OW

By using continuous compounding of inflation and discount rate, the present worth of the various costs during the cycle $(0, t_r)$ is evaluated as follows.

(a) Present worth of the inventory holding cost in the RW is

$$= \int_o^{t_r} (F+\gamma t)e^{-Rt} I_r(t)dt$$

$$= F\int_o^{t_r} e^{-Rt}\left[\frac{-ae^{bt}}{b+\beta}+\frac{ae^{(b+\beta)t_r}-e^{-\beta t}}{b+\beta}\right]dt+\gamma\int_o^{t_r} te^{-Rt}\left[\frac{-ae^{bt}}{b+\beta}+\frac{ae^{(b+\beta)t_r}-e^{-\beta t}}{b+\beta}\right]dt$$

$$H.C.=\frac{Fa}{(b+\beta)}\left[\frac{1}{b-R}-\frac{e^{(b-R)t_r}(b+\beta)}{(b-R)(\beta+R)}+\frac{e^{(b+\beta)t_r}}{R+\beta}\right]-\frac{\gamma a t_r e^{(b-R)t_r}}{(b+\beta)(b-R)}+\frac{\gamma a e^{(b-R)t_r}}{(b+\beta)(b-R)^2}$$
$$+\frac{\gamma a e^{(b-R)t_r}}{(b+\beta)}\left[\frac{1}{(b-R)^2}-\frac{1}{(R+\beta)}-\frac{1}{(R+\beta)^2}\right]+\frac{\gamma a e^{(b+\beta)t_r}}{(b+\beta)(R+\beta)^2} \quad \text{.... (15)}$$

(b) Present worth of the inventory holding cost in the OW is

$$= \int_o^{t_o} (H+\gamma' t)e^{-Rt} I_o(t)dt$$

$$= H\int_o^{t_o} e^{-Rt}\left[-\frac{ae^{bt}}{b+\alpha}+\frac{ae^{(b+\alpha)t_o}}{b+\alpha}e^{-\alpha t}+We^{-\alpha t}\right]dt+\gamma'\int_o^{t_o} te^{-Rt}\left[-\frac{ae^{bt}}{b+\alpha}+\frac{ae^{(b+\alpha)t_o}}{b+\alpha}e^{-\alpha t}+We^{-\alpha t}\right]dt$$

$$= H\int_o^{t_r} We^{-(\alpha+R)t}dt + H\int_{t_r}^{t_0}\left(-\frac{a}{b+\alpha}e^{(b-R)t}+\frac{a}{b+\alpha}e^{(b+\alpha)t_o}e^{-(R+\alpha)t}\right)dt$$
$$+\gamma'\int_o^{t_r} Wte^{-(\alpha+R)t}dt+\gamma'\int_{t_r}^{t_0} t\left(-\frac{a}{b+\alpha}e^{(b-R)t}+\frac{a}{b+\alpha}e^{(b+\alpha)t_o}e^{-(R+\alpha)t}\right)dt$$

$$= H\left[\frac{W}{(R+\alpha)}-\frac{We^{-(R+\alpha)t_r}}{(R+\alpha)}-\frac{ae^{(b-R)t_0}}{(b-R)(R+\alpha)}+\frac{ae^{(b-R)tr}}{(b+\alpha)(b-R)}+\frac{ae^{(b+\alpha)t_0}e^{-(R+\alpha)tr}}{(b+\alpha)(R+\alpha)}\right]$$
$$+\frac{\gamma' W}{(R+\alpha)^2}-\frac{W\gamma' t_r e^{-(R+\alpha)t_r}}{(R+\alpha)}-\frac{W\gamma' e^{-(R+\alpha)t_r}}{(R+\alpha)^2}-\frac{\gamma' t_0 a e^{(b-R)t_0}}{(b+\alpha)(b-R)}+\frac{\gamma' t_r a e^{(b-R)t_r}}{(b+\alpha)(b-R)}$$
$$+\frac{\gamma' a e^{(b-R)t_0}}{(b+\alpha)}\left[\frac{1}{(b-R)^2}-\frac{1}{(R+\alpha)^2}\right]-\frac{\gamma' t_0 a e^{(b-R)t_0}}{(b+\alpha)(R+\alpha)}+\frac{\gamma' t_r a e^{-(R+\alpha)t_r}e^{(b+\alpha)t_0}}{(b+\alpha)(R+\alpha)}$$
$$+\frac{\gamma' a e^{-(R+\alpha)t_r}e^{(b+\alpha)t_0}}{(b+\alpha)(R+\alpha)^2}-\frac{\gamma' a e^{(b-R)t_r}}{(b-R)^2(b+\alpha)}$$

.... (16)

(c) Present worth of the replenishment cost is = A1

(d) Present worth of the backlogging cost is $S\int_{to}^{t_s} B(t)e^{-Rt}dt$

$$= S\int_{t_o}^{t_s} e^{-Rt}\left[\frac{a}{b+\delta}e^{-\delta t_s}\left(e^{(b+\delta)t} - e^{(b+\delta)t_o}\right)\right]dt$$

$$= S\int_{t_o}^{t_s}\left(e^{-Rt}\frac{a}{b+\delta}e^{-\delta t_s}e^{(b+\delta)t} - \frac{a}{b+\delta}e^{(b+\delta)t_o}e^{-\delta t_s}e^{-Rt}\right)dt$$

$$\frac{Sae^{-\delta t_s}}{(b+\delta)(b+\delta-R)}\left(e^{(b+\delta-R)t_s} - e^{(b+\delta-R)t_o}\right) + \frac{Sae^{(b+\delta)t_o}e^{-\delta t_s}}{(b+\delta)R}[e^{-Rt_s} - e^{-Rt_o}] \quad \text{.... (17)}$$

(e) Present worth of the opportunity cost due to cost sales is

$$= c_1 e^{-Rt_s}\int_{t_o}^{t_s}(1-e^{-\delta(t_s-1)})ae^{bt}dt$$

$$= c_1 e^{-Rt_s}\left[\frac{a}{b}\left(e^{bt} - e^{bt_o}\right) - \frac{a}{b+\delta}e^{-\delta t_s}\left(e^{(b+\delta)t} - e^{(b+\delta)t_o}\right)\right] \quad \text{.... (18)}$$

Now the amounts of deteriorated items in both the RW and OW during $(0,\, t_o)$ are

$$\beta\int_o^{t_r} I_r(t)dt \text{ and } \alpha\int_o^{t_o} I_o(t)dt$$

(f) Therefore the present worth of the cost for the deteriorated items is

$$C\left[\beta\int_o^{t_r} e^{-Rt}I_r(t)dt + \alpha\int_o^{t_r} e^{-Rt}I_o(t)dt\right] \quad \text{.... (19)}$$

Now, the present worth of the total relevant cost per unit time during the cycle $(0,\, t_s)$ using equation (15)-(19) is given by.

$$TC2(t_r, t_s) = \frac{1}{t_s}\left[A1 + \frac{(F+C\beta)a}{b+\beta}\left\{\frac{1}{b-R} - \frac{e^{(b-R)t_r}(b+\beta)}{(b-R)(R+\beta)} + \frac{e^{(b+\beta)t_r}}{R+\beta}\right\} - \frac{\gamma a t_r e^{(b-R)t_r}}{(b-R)(b+\beta)}\right.$$

$$+\frac{\gamma a e^{(b-R)t_r}}{(b+\beta)}\left[\frac{1}{(b-R)^2} - \frac{(R+\beta+1)}{(R+\beta)^2}\right] - \frac{\gamma a}{(b+\beta)(b-R)^2} + \frac{\gamma a e^{(b+\beta)t_r}}{(R+\beta)^2(b+\beta)}$$

$$+(H+c\alpha)\left(\frac{W}{(R+\alpha)} - \frac{We^{-(R+\alpha)t_r}}{\alpha+R} - \frac{ae^{(b-R)t_o}}{(R+\alpha)(b-R)} + \frac{ae^{(b-R)t_r}}{(b+\alpha)(b-R)} + \frac{ae^{(b+\alpha)t_0}e^{-(R+\alpha)t_r}}{(b+\alpha)(R+\alpha)}\right)$$

$$+\frac{\gamma' W}{(R+\alpha)^2} - \frac{\gamma' t_r We^{-(R+\alpha)t_r}}{\alpha+R} - \frac{\gamma' We^{-(R+\alpha)t_r}}{(\alpha+R)^2} - \frac{a\gamma' t_0 e^{(b-R)t_o}}{(R+\alpha)(b-R)} + \frac{a\gamma' t_r e^{(b-R)t_r}}{(b+\alpha)(b-R)}$$

$$+\frac{a\gamma' e^{(b-R)t_0}}{(b+\alpha)}\left[\frac{1}{(b-R)^2} - \frac{1}{(R+\alpha)^2}\right] + \frac{a\gamma' t_r e^{-(R+\alpha)t_r}e^{(b+\alpha)t_0}}{(b+\alpha)(R+\alpha)} + \frac{a\gamma' e^{-(R+\alpha)t_r}e^{(b+\alpha)t_0}}{(b+\alpha)(R+\alpha)^2} - \frac{a\gamma' e^{(b-R)t_r}}{(b+\alpha)(b-R)^2}$$

$$+\frac{Sae^{-\delta t_s}}{(b+\delta)(b+\delta-R)}\left(e^{(b+\delta-R)t_s} - e^{(b+\delta-R)t_0}\right) + \frac{Sa}{(b+\delta)}\frac{e^{-\delta t_s}e^{(b+\delta)t_0}}{R}\left(e^{-Rt_s} - e^{-Rt_0}\right)$$

$$\left.+c_1 e^{-Rt_s}\left(\frac{a}{b}(e^{bt} - e^{bt_o}) - \frac{a}{b+\delta}e^{-\delta t_s}(e^{(b+\delta)t} - e^{(b+\delta)t_o})\right)\right]$$

$$\frac{\partial TC2(t_r,t_s)}{\partial t_r}=\frac{1}{t_s}\left\{\frac{(F+c\beta)ae^{(b+\beta)t_r}}{(R+\beta)}-\frac{(F+c\beta)ae^{(b-R)t_r}}{(R+\beta)}-\frac{\gamma at_re^{(b-R)t_r}}{(b+\beta)}\right.$$

$$-\frac{\gamma a(b-R)(R+\beta+1)e^{(b-R)t_r}}{(b+\beta)(R+\beta)}+\frac{\gamma ae^{(b+\beta)t_r}}{(R+\beta)^2}$$

$$+(H+c\alpha)\left[We^{-(R+\alpha)t_r}+\frac{ae^{(b-R)t_r}}{b+\alpha}-\frac{ae^{-(R+\alpha)t_r}e^{(b+\alpha)t_0}}{(b+\alpha)}\right]+\frac{(H+c\alpha)a^2e^{(b+\alpha)t_r}}{(R+\alpha)\left((b+\alpha)W+ae^{(b+\alpha)t_r}\right)}$$

$$\left[e^{-(R+\alpha)t_r}e^{(b+\alpha)t_0}-e^{(b-R)t_0}\right]+W\gamma' t_re^{-(R+\alpha)t_r}+\frac{\gamma' t_rae^{(b-R)t_r}}{b+\alpha}+\frac{\gamma' t_rae^{-(R+\alpha)t_r}e^{(b+\alpha)t_0}}{b+\alpha}$$

$$+\left[\frac{\gamma' e^{(b-R)t_0}}{(b+\alpha)(R+\alpha)}-\frac{\gamma' t_0e^{(b-R)t_0}}{(R+\alpha)}-\frac{\gamma'(b-R)e^{(b-R)t_0}}{(b+\alpha)(R+\alpha)^2}+\frac{\gamma' e^{-(R+\alpha)t_r}e^{(b+\alpha)t_0}}{(R+\alpha)^2}+\frac{\gamma' t_re^{-(R+\alpha)t_r}e^{(b+\alpha)t_0}}{(R+\alpha)}\right]$$

$$\times\frac{a^2e^{(b+\alpha)t_r}}{(b+\alpha)W+ae^{(b+\alpha)t_r}}+\frac{Sa^2e^{(b+\alpha)t_r}}{(b+\alpha)W+ae^{(b+\alpha)t_r}}\left[\frac{e^{-(R+\delta)t_s}e^{(b+\delta)t_o}}{R}-\frac{e^{(b+\delta-R)t_o}e^{-\delta t_s}}{b+\delta}\right.$$

$$\left.\left.-\frac{(b+\delta-R)e^{-\delta t_s}e^{(b+\delta-R)t_o}}{(b+\delta)R}\right]+\frac{c_1a^2e^{(b+\alpha)t_r}}{(b+\alpha)W+ae^{(b+\alpha)t_r}}\left(e^{(b+\delta)t_o}e^{-(R+\delta)t_s}-e^{-Rt_s}e^{bt_0}\right)\right\}$$

Now $$\frac{\partial TC2(t_r,t_s)}{\partial t_s}=\frac{1}{t_s}\left[0+0+0+\frac{Sa(b-R)e^{(b-R)t_s}}{(b+\delta)(b+\delta-R)}+\frac{Sa\delta e^{-\delta t_s}e^{(b+\delta-R)t_0}}{(b+\delta)(b+\delta-R)}\right.$$

$$-\frac{Sa(\delta+R)e^{(b+\delta)t_0}e^{-(\delta+R)t_s}}{(b+\delta)R}+\frac{Sa\delta e^{(b+\delta-R)t_0}e^{-\delta t_s}}{(b+\delta)R}-\frac{Rc_1ae^{-Rt_s}}{b}[e^{bt}-e^{bt_0}]$$

$$\left.+\frac{c_1a(R+\delta)e^{-(R+\delta)t_s}}{(b+\delta)}\left(e^{(b+\delta)t}-e^{(b+\delta)t_o}\right)\right]$$

$$-\frac{1}{t_s^2}\left[A1+\frac{(F+C\beta)a}{b+\beta}\left\{\frac{1}{b-R}-\frac{e^{(b-R)t_r}(b+\beta)}{(b-R)(R+\beta)}+\frac{e^{(b+\beta)t_r}}{R+\beta}\right\}-\frac{\gamma at_re^{(b-R)t_r}}{(b-R)(b+\beta)}\right.$$

$$+\frac{\gamma ae^{(b-R)t_r}}{(b+\beta)}\left[\frac{1}{(b-R)^2}-\frac{(R+\beta+1)}{(R+\beta)^2}\right]-\frac{\gamma a}{(b+\beta)(b-R)^2}+\frac{\gamma ae^{(b+\beta)t_r}}{(R+\beta)^2(b+\beta)}$$

$$+(H+c\alpha)\left(\frac{W}{(R+\alpha)}-\frac{We^{-(R+\alpha)t_r}}{\alpha+R}-\frac{ae^{(b-R)t_o}}{(R+\alpha)(b-R)}+\frac{ae^{(b-R)t_r}}{(b+\alpha)(b-R)}+\frac{ae^{(b+\alpha)t_0}e^{-(R+\alpha)t_r}}{(b+\alpha)(R+\alpha)}\right)$$

$$+\frac{\gamma' W}{(R+\alpha)^2}-\frac{\gamma' t_r We^{-(R+\alpha)t_r}}{\alpha+R}-\frac{\gamma' We^{-(R+\alpha)t_r}}{(\alpha+R)^2}-\frac{a\gamma' t_0 e^{(b-R)t_o}}{(R+\alpha)(b-R)}+\frac{a\gamma' t_r e^{(b-R)t_r}}{(b+\alpha)(b-R)}$$

$$+\frac{a\gamma' e^{(b-R)t_0}}{(b+\alpha)}\left[\frac{1}{(b-R)^2}-\frac{1}{(R+\alpha)^2}\right]+\frac{a\gamma' t_r e^{-(R+\alpha)t_r}e^{(b+\alpha)t_0}}{(b+\alpha)(R+\alpha)}+\frac{a\gamma' e^{-(R+\alpha)t_r}e^{(b+\alpha)t_0}}{(b+\alpha)(R+\alpha)^2}-\frac{a\gamma' e^{(b-R)t_r}}{(b+\alpha)(b-R)^2}$$

$$+c_1e^{-Rt_s}\left(\frac{a}{b}(e^{bt}-e^{bt_o})-\frac{a}{b+\delta}e^{-\delta t_s}(e^{(b+\delta)t}-e^{(b+\delta)t_o})\right)\Bigg]$$

The optimal values of $t_r$ and $t_s$ for minimum present value of the total relevant cost per unit time is any solution of the system.

$$\frac{\partial\partial TC2(t_r,t_s)}{\partial t_r}=0 \text{ and } \frac{\partial\partial TC2(t_r,t_s)}{\partial t_s}=0$$

Which also satisfies the conditions

$$\frac{\partial^2 TC2(t_r,t_s)}{\partial t_r^2}>0\ ,\quad \frac{\partial^2 TC2(t_r,t_s)}{\partial t_s^2}>0 \text{ and}$$

$$\left(\frac{\partial^2 TC2(t_r,t_s)}{\partial t_r^2}\right)\left(\frac{\partial^2 TC2(t_r,t_s)}{\partial t_s^2}\right)-\left(\frac{\partial^2 TC2(t_r,t_s)}{\partial t_r \partial t_s}\right)>0$$

Using these optimal values of $t_r$ and $t_s$ the optimal values of Z, Q and the minimum average cost can be obtained from (10), (11) and (20), respectively.

### 4. Single-storage model

At time t=0, a lot size of Q units enters the system from which a portion is used to meet the partial backlogged items towards previous shortages, and the initial inventory for the period is Z. By the time tr, the inventory level reaches zero due to the combined effect of demand and deterioration and thereafter the shortages begin to accumulate and continue up to ts. The partially backlogged quantity is supplied to the customers at the beginning of the next cycle. As described in the two-warehousing section, the present value of the total relevant cost per unit time can be obtained as follows:

$$TC1(t_r,\,t_s)=\frac{1}{t_s}\left[A+\frac{(H+c\alpha)a}{b+\alpha}\left\{\frac{1}{b-R}-\frac{e^{(b-R)t_r}(b+\alpha)}{(b-R)(R+\alpha)}+\frac{e^{(b+\alpha)t_r}}{R+\alpha}\right\}-\frac{t_0 a\lambda e^{(b-R)t_r}}{(b-R)(b+\alpha)}\right.$$

$$+\frac{a\lambda e^{(b-R)t_r}}{(b+\alpha)}\left[\frac{1}{(b-R)^2}-\frac{(R+\alpha+1)}{(R+\alpha)^2}\right]-\frac{a\lambda}{(b+\alpha)(b-R)^2}+\frac{a\lambda e^{(b+\alpha)t_r}}{(b+\alpha)(R+\alpha)^2}$$

$$+\frac{Sae^{-\delta t_s}}{(b+\delta)(b+\delta-R)}\left(e^{(b+\delta-R)t_s}-e^{(b+\delta-R)t_r}\right)+\frac{Sa}{(b+\delta)}\frac{e^{-\delta t_s}e^{(b+\delta)t_r}}{R}\left(e^{-Rt_s}-e^{-Rt_r}\right)$$

$$\left.+c_1e^{-Rt_s}\left(\frac{a}{b}(e^{bt}-e^{bt_r})-\frac{a}{b+\delta}e^{-\delta t_s}(e^{(b+\delta)t}-e^{(b+\delta)t_r})\right)\right] \qquad (21)$$

Where $$Z=\frac{ae^{(b+\alpha)t_r}}{b+\alpha}-\frac{a}{b+\alpha} \qquad (22)$$ and $$Q=Z+\frac{a}{b+\delta}e^{-\delta t_s}\left[e^{(b+\delta)t_s}-e^{(b+\delta)t_r}\right] \qquad (23)$$

Where λ is a holding cost parameter.

The optimal values of $t_r$ and $t_s$ for minimum present value of the total relevant cost per unit time is any solution of the system

$$\frac{\partial TC1(t_r,t_s)}{\partial t_r}=\frac{1}{t_s}\left\{\frac{(H+c\alpha)ae^{(b+\alpha)t_r}}{R+\alpha}-\frac{(H+c\alpha)ae^{(b-R)t_r}}{R+\alpha}+\frac{\lambda ae^{(b+\alpha)t_r}}{(R+\alpha)^2}\right.$$
$$-\frac{\gamma a(b-R)(R+\alpha+1)e^{(b+\alpha)t_r}}{(R+\alpha)^2(b+\alpha)}-\frac{\lambda at_re^{(b-R)t_r}}{b+\alpha}-\frac{Sae^{-\delta t_s}e^{(b+\delta-R)t_r}}{b+\delta}$$
$$\left.+\frac{Sae^{-(R+\delta)t_s}e^{(b+\delta)t_r}}{R}-\frac{Sa(b+\delta-R)e^{(b+\delta-R)t_r}e^{-\delta t_s}}{(b+\delta)R}-C_1ae^{-Rt_s}e^{bt_r}+ae^{(b+\delta)t_r}e^{-\delta t_s}\right]$$

and

$$\frac{\partial TC1(t_r,t_s)}{\partial t_s}=\frac{1}{t_s}\left[\frac{Sa\delta e^{-\delta t_s}e^{(b+\delta-R)t_r}}{b+\delta}-\frac{Sa(R+\delta)e^{-(R+\delta)t_s}e^{(b+\delta)t_r}}{R}\right.$$

$$\left.+\frac{Sa\delta(b+\delta-R)e^{-\delta t_s}e^{(b+\delta-R)t_r}}{(b+\delta)R}+C_1a\,\mathrm{Re}^{-Rt_s}e^{bt_r}-a\delta e^{-\delta t_s}e^{(b+\delta)t_r}\right]$$
$$-\frac{1}{t_s^2}\left[A+\frac{(H+c\alpha)a}{b+\alpha}\left\{\frac{1}{b-R}-\frac{e^{(b-R)t_r}(b+\alpha)}{(b-R)(R+\alpha)}+\frac{e^{(b+\alpha)t_r}}{R+\alpha}\right\}-\frac{t_0a\lambda e^{(b-R)t_r}}{(b-R)(b+\alpha)}\right.$$
$$+\frac{a\lambda e^{(b-R)t_r}}{(b+\alpha)}\left[\frac{1}{(b-R)^2}-\frac{(R+\alpha+1)}{(R+\alpha)^2}\right]-\frac{a\lambda}{(b+\alpha)(b-R)^2}+\frac{a\lambda e^{(b+\alpha)t_r}}{(b+\alpha)(R+\alpha)^2}$$

$$+\frac{Sae^{-\delta t_s}}{(b+\delta)(b+\delta-R)}\left(e^{(b+\delta-R)t_s}-e^{(b+\delta-R)t_r}\right)+\frac{Sa}{(b+\delta)}\frac{e^{-\delta t_s}e^{(b+\delta)t_r}}{R}\left(e^{-Rt_s}-e^{-Rt_r}\right)$$
$$\left.+c_1e^{-Rt_s}\left(\frac{a}{b}(e^{bt}-e^{bt_r})-\frac{a}{b+\delta}e^{-\delta t_s}(e^{(b+\delta)t}-e^{(b+\delta)t_r})\right)\right]$$

Now $\frac{\partial\partial TC1(t_r,t_s)}{\partial t_r}=0$ and $\frac{\partial\partial TC1(t_r,t_s)}{\partial t_s}=0$

Which also satisfies the conditions

$$\frac{\partial^2TC1(t_r,t_s)}{\partial t_r^2}>0\ ,\quad \frac{\partial^2TC1(t_r,t_s)}{\partial t_s^2}>0 \text{ and}$$

$$\left(\frac{\partial^2TC1(t_r,t_s)}{\partial t_r^2}\right)\left(\frac{\partial^2TC1(t_r,t_s)}{\partial t_s^2}\right)-\left(\frac{\partial^2TC1(t_r,t_s)}{\partial t_r\partial t_s}\right)>0$$

Using these optimal values of $t_r$ and $t_s$ the optimal values of Z, Q and the minimum average cost can be obtained from (21), (22) and (23), respectively.

## 1. Numerical example

In this section, we assume that the deterioration rate in the RW is greater than that of the OW. Secondly, it is the opposite, that is, the deterioration rate in the RW is smaller than that of the OW. Under these conditions, the total cost has been computed for different values of demand parameters (a and b) for the single- as well as the two-warehouse system. Now, depending on the total costs computed from both the systems, a decision is made whether to rent a warehouse or not.

Example 1: Let a = 290, b=4, c=9, H=0.9, F=2.4, A=80, A1=100, s=4, c1=11, δ=0.80, W=90, γ=0.01 , $\gamma'=0.001$, λ=0.002, α=0.02, β=0.04, R=0.05 in appropriate units. Solving the single- and two-warehouse models we get TC1=325.7 and TC2=297.6 As TC2<TC1, therefore it is beneficial to use a rented warehouse. The corresponding values of the two-warehouse model are $t_r$ =0.19, $t_s$=0.58, Z=157.5 Q=178.5

Example 2: Let a=190, b=8, c=9, H=0.9, F=2.4, A=80, A1=100, s=4, c1=11, δ=0.80, γ=0.01, $\gamma'=0.001,\ \lambda=0.002,$ W=90,α=0.02, β=0.04, R=0.05 in appropriate units. Solving the single- and two-warehouse models we get TC1=239.5 and TC2=241.8. Since TC2>TC1, therefore the decision maker should not rent a warehouse. The corresponding values of single-warehouse model are $t_r$=0.47, $t_s$=0.55, Z=98.0, Q=114.7.

## II. CONCLUSION

The proposed model incorporates some realistic and practical features that are likely to be associated with the inventory of certain types of goods, such as: demand rate is time dependent, the inventory deteriorates at a variable rate over time and production rate is flexible and considers two warehouses to reflect realistic business situations. Most products experience a period of rapid demand increase during the introduction phase of product life .The present model differs from the existing models, as here exponentially increasing demand has been considered in place of constant demand; since the exponentially increasing demand is very much applicable to a lot of consumer goods whose demand changes steadily along with a steady increase in population density. In addition of this exponentially increase demand we have assumed that holding cost is variable not a constant .Moreover, it is seen that some but not all the customers will wait for backlogged items during a shortage period, thus shortages at the owned warehouse are also allowed subject to partial backlogging. A solution procedure has also been presented, which helps the decision-maker to decide under what circumstances he has to rent a warehouse. Findings have been validated with the help of some numerical examples, and sensitivity analysis of the optimal solution with respect to various parameters has also been presented.

## REFERENCES


1. **Abad, P.L. (1996).** "Optimal pricing and lot sizing under conditions of perishability and partial backordering", *Management Science; 42: 1093-1104.*
2. **Abad, P.L. (2000).** "Optimal lot size for a perishable good under conditions of finite production and partial backordering and lost sale". *Computers & Industrial Engineering; 38(4): 457-465.*
3. **Abad, P.L. (2001).** "Optimal price and order size for a reseller under partial backordering", *Computers and Operations Research; 28: 53-65.*
4. **Balkhi, Z.T., Benkherouf, L. (1996).** "A production lot size inventory model for deteriorating items and arbitrary production and demand rate", *European Journal of Operational Research; 92: 302–309.*
5. **Bose, S., Goswami, A., Chaudhuri, K.S. (1995).** "An EOQ model for deteriorating items with linear time-dependent demand rate and shortages under inflation and time discounting", *Journal of the Opertaional research Society; 46:771-782.*
6. **Buzacott, J.A. (1975).** "Economic order quantity with inflation", *Operations Research Qtr; (3): 553-558.*
7. **Chang, C.T. (2004).** "An EOQ model with deteriorating items under inflation when supplier credits linked to order quantities", *International Journal of Production Economics; 88: 307-316.*
8. **Goyal, S.K., Giri, B.C. (2003).** "The production-inventory problem of a product with time varying demand, production and deterioration rates", *European Journal of Operational Research; 147: 549-557.*
9. **Jaggi, C.K., Aggarwal, K.K., Goel, S.K. (2006).** "Optimal order policy for deteriorating items with inflation induced demand", *International Journal of Production Economics; 103: 707-714.*
10. **Jolai, F., Tavakkoli-Moghaddam, R., Rabbani, M., Sadoughian, M.R. (2006).** "An economic production lot size model with deteriorating items, stock-dependent demand, inflation, and partial backlogging", *Applied Mathematics and Computation; 181: 380-389.*
11. **Li, R., Lan, H., Mawhinney, J. R. (2010).** "A Review on deteriorating inventory study", *J. Service Science & Management; 3:117-129.*
12. **Misra, R.B. (1975).** "Optimum production lot size model for a system with deteriorating inventory", *International Journal of Production Research; 13: 495–505.*
13. **Ouyang, L.Y., Wu, K.S., Cheng, M.C. (2005). "**An inventory model for deteriorating items with exponential declining demand and partial backlogging", *Yugoslav Journal of Operations Research; 15(2): 277-288.*
14. **Papachristos, S., Skouri, K. (2000).** "An optimal replenishment policy for deteriorating items with time-varying demand and partial-exponential type backlogging", *Operations Research Letters; 27(4): 175-184..*
15. **Wee, H.M., Law, S.T. (1999).** "Economic production lot size for deteriorating items taking account of time value of money", *Computers and Opertaional Research; 26: 545-558.*
16. **Yang, G.K. (2007). "**Note on sensitivity analysis of inventory model with partial backorders", *European Journal of Operational Research; 177: 865-871.*