International Journal of Soft Computing and Engineering (IJSCE)
ISSN: 2231-2307, Volume-2 Issue-4, September 2012

Scattered Domination in Graphs

Rajesh Kumar Tripathi, Minakshi Gaur

Abstract- ***In this paper, we want to compute an optimal scattered domination for domination graph. In this paper, we also show optimal broadcast domination is in path P. We first prove that every graph has an optimal scattered domination in which the subset of vertices dominated by the same vertex is ordered in a path or a cycle. Using this, we give a polynomial time algorithm for computing optimal broadcast domination of arbitrary graphs.***

Keywords- ***Domination graph, scattered domination graph, path, broadcast domination of graph.***

I. INTRODUCTION

A dominating set in a graph is a subset of the vertices of the graph such that every vertex of the graph either belongs to the dominating set or has a neighbor in the dominating set .A vertex outside of the dominating set is said to be dominated by one of its neighbors in the dominating set. The standard optimal domination problem seeks to find a dominating set of minimum cardinality. Since the introduction of this problem [2],[1],many domination related parameters have been introduced and studied , and domination in graphs is one of the most well known and widely studied subjects within graph algorithms[7]. In this paper, we show that, quite surprisingly, optimal broadcast domination is in path P. We first prove that every graph has an optimal scattered domination in which the subset of vertices dominated by the same vertex is ordered in a path or a cycle. Using this, we give a polynomial time algorithm for computing optimal broadcast domination of arbitrary graphs. Our algorithm computes minimum weight path in an auxiliary graph , and thus differs from standard methods of proving polynomial time bounds, like reductions to 2-SAT or 2-dimensional matching.

II. DEFINITIONS AND TERMINOLOGY

In this chapter we work with unweighted, undirected, connected, and simple graphs as input graphs to our problem. Let G = (V, E) be a graph with $n = |V|$ and $m = |E|$. For any vertex $v \in V$ the neighborhood of v is the set $N_G(v) = \{u : uv \in E\}$. Similarly, for any set $S \subseteq V, N_G(S) = \bigcup_{v\in S} N(v) - S$. We let G(S) denote the sub graph of G induced by S.The distance between two vertices u and v in G, denoted by $d_G(u,v)$, is the minimum number of edges on a path between u and v.

Manuscript received September 02, 2012.
Rajesh Kumar Tripathi, Astt Professor Deptt. of Mathematics, Bharat Institute Technology, Meerut(U.P.), India
Dr.Minakshi Gaur, Head, Depts. Mathematics, N.A.S.(PG) College, Meerut(U.P.), India

The eccentricity of a vertex v, denoted by e (v), is the largest distance from v to any vertex of G. The radius of G, denoted by rad (G), is smallest eccentricity in G. The diameter of G, denoted by diam (G), is the largest distance between any pair of vertices in G.

A function $f : V \rightarrow \{0,1,........diam(G)\}$ is a scattered on G. The set of scattered dominators defined by f is the set $V_f = \{v \in V : f(v) \geq 1\}$. A broadcast is dominating if for every vertex $u \in V$ there is a vertex $v \in V_f$ such that $d(u,v) \leq f(v)$. In this case f is also called a scattered domination. The cost of scattered f incurred by a set $S \subseteq V$ is $c_f(S) = \sum_{v\in S} f(v)$. Thus, $c_f(V)$ is the total cost incurred by broadcast function f on G.For a vertex $v \in V$ and an integer $p \geq 1$, we define the ball $B_G(v,p)$ to be the set of vertices that are the distance $\leq p$ from v in G. Thus $B_G(v, f(v))$ is the set of all vertices that are dominated by v(including v itself) if $f(v) \geq 1$. We will omit the subscript G in the notation for balls, since a ball will always refer to the input graph G. A scattered domination f on G is efficient if $B(u, f(u)) \cap B(v, f(v)) = \phi$ for all pairs of distinct vertices $u, v \in V$ .For an efficient scattered domination f on G, we define the domination graph $G_f = (V_f\{uv : N_G(B(u, f(u))) \cap B(v, f(v)) \neq \phi\})$. Hence the domination can be seen as a modification of G in which every ball $B\ (v, f(v))$ is contracted to the single vertex v, and neighborhoods are preserved. Since G is connected and f is dominating, G_f is always connected. An example is given in Figure1

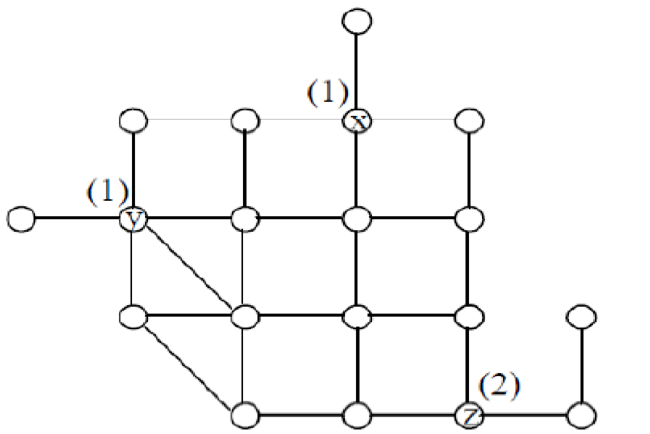

Fig.1.On the left hand side, a graph G with an efficient scattered domination f is shown.

Published By:
Blue Eyes Intelligence Engineering
& Sciences Publication

The optimal scattered domination problem on a given graph G asks to compute domination on G with the minimum cost. Note that if f is an optimal scattered domination on G = (V,E) , then $c_f(V) \le rad(G)$ since one can always choose a vertex v of smallest eccentricity and dominate all other vertices with $f(v) = e(v) = rad(G). if\ c_f(V) = rad(G) = f(v)$ for a single vertex v in G, then f is called a radial scattered domination.

We now add the following results.

In [4], Dunbar shows that every graph has an optimal scattered domination that is efficient. In particular, the following lemma is implicit from the proof of this result.

Lemma1. (Dunbar et al. [4]) For any non efficient scattered domination f on a graph G = (V, E), there is an efficient scattered domination f' on G such that $\left|V_{f'}\right| < \left|V_f\right|\ and\ c_{f'}(V) = c_f(V)$

Lemma 2. Let f be an efficient scattered domination on G = (V, E). If the domination graph G_f has vertex of degree >2, then there is an efficient scattered domination f' on G such that $\left|V_{f'}\right| < \left|V_f\right|\ and\ c_{f'}(V) = c_f(V)$.

Proof. Let v be a vertex with degree >2 in G_f, and let x,y , and z be three of the neighbors of v in G_f. By the way the domination graph G_f is defined, v, x , y , and z are also vertices in G , and they all have scattered powers ≥ 1 in f . Since f is efficient, $d_G(v,x) = f(v) + f(x) + 1.\ similarly, d_G(v,y) = f(v) + f(y) + 1\ and\ d_G(v,z) = f(v) + f(z) + 1$. Assume without loss of generality that $f(x) \le f(y) \le f(z)$. If $f(x) + f(y) > f(z)$ then we construct a new f' on G with

$f'(u) = f(u)\ \ for\ all\ vertices\ u \in V \setminus \{v,x,y,z\}$.Further more, we let

$$f'(v) = f(v) + f(x) + f(y) + f(z),$$
$$and\ \ f'(x) = f'(y) = f'(z) = 0$$

The new f' is dominating since every vertex that was previously dominated by one of v, x, y, or z now dominated by v. Thus $d_G(u,v) \le f(v) + 2f(z) + 1$ by our assump- tions. Since $f'(v) > f(v) + 2f(z)$, vertex u now dominated by v in f'. The cost of f' is the same as that of f, and the number of dominators in f' is smaller.Let now $f(x) + f(y) \le f(z)$.As we mentioned above, there is a path P in G between v and z of length $f(v) + f(z) + 1$. Let w be a vertex on P such that the number of edges between w and z on P is $f(v) + f(x) + f(y)$. Since f is efficient, f (w) = 0. We construct a new f' on G such that $f'(u) = f(u)\ \ for\ all\ vertices\ u \in V \setminus \{v,w,x,y,z\}$. Furthermore, we let.

$$f'(w) = f(v) + f(x) + f(y) + f(z),$$
$$and\ f'(v) = f'(x) = f'(y) = f'(z) = 0$$

By the way $d_G(z,w)$ is defined, any vertex that was dominated by z or v in f is now dominated by w, since $d_G(v,w) < f(z)$. Let u be the vertex that was dominated by y in f. The distance between u and

$$w\ in\ G\ is \le 2f(y) + 2f(v) + f(z) + 2 - f(v)$$
$$- f(x) - f(y) = f(y) + f(v) + f(z) + 2 - f(x)$$
$$\le f(y) + f(v) + f(z) + f(x) = f'(w).$$

Thus u is now dominated by w.The same is true for any vertex that was dominated by x in f since we assumed that $f(x) \le f(y)$. Thus f' is domination. Clearly, the cost of f' and f are the same, and f' has fewer dominators.

Thus we have shown how to compute a new domination f' as desired. If f' is not efficient, then by Lemma 1 there exists an efficient domination with the same cost and fewer dominators, so the lemma follows.

We now ready to state the main result of this section, on which our algorithm will be based.

Theorem 1. For any graph G, there is an efficient optimal domination f on G , such that the domination graph G_f is either a path or a cycle.

Proof. Let f be any efficient optimal domination on G = (V,E) . If G_f has a vertex of degree>2 then by the Lemma 2, an efficient domination f' on G with $\left|V_{f'}\right| < \left|V_f\right|\ and\ c_{f'}(V) = c_f(V)$ exists. The proofs of both lemmas 1 and 2 are constructive, so we know how to obtain f'. As long as there are vertices of degree >2 in the domination graph, this process can be repeated. Since we always obtain a new domination graph with a strictly smaller number of vertices, the process has to stop after <n steps. Since domination graphs are connected, the theorem follows.

Corollary 1. For any graph G = (V,E) , there is an efficient optimal scattered domination f on G such that removing the vertices of B(v,f(v)) from GF results in at most two connected components , for every $v \in V_f$.

Corollary 2. For any graph G = (V,E) , there is an efficient optimal domination f on G such that $x \in V_f$ satisfies the following : $G' = G(V \setminus B(x, f(x)))$ is connected (or empty), and G' has an efficient optimal domination $f'\ \ such\ that\ G'_{f'}$ is a path (or empty).

Computing an Optimal Scattered Domination

By Theorem 1 we know that an efficient optimal f on G must exist such that G_f is a path or a cycle. We will first give an algorithm for handling the case when G_f is a path.

III. OPTIMAL SCATTERED DOMINATION WHEN THE DOMINATION GRAPH IS A PATH

In this section, we want to find an efficient domination of minimum cost over all dominations f on G = (V, E) such that G_f is a path. Our approach will be as follows: for each vertex u of G, we will compute a new graph G_u, and use this to find the best possible domination f such that G_f is a path and u belongs to a ball corresponding to one of the endpoints of G_f. We will repeat this process for every u in G, and choose at the end the best f ever computed.

Given a vertex $u \in V$, we define a directed graph G_u with weights assigned to its vertices as follows: For each $v \in V$ and each $p \in [1,..,rad(G)]$, there is a vertex (v, p) in G_u if and only if one of the following is true :

- $G(V \setminus B(v,p))$ is connected or empty and $u \in B(v,p)$
- $G(V \setminus B(v,p))$ has at most two connected components and $u \notin B(v,p)$.

Thus G_u have a total of at most n.rad (G) vertices. Following corollaries 1 and 2, each vertex (v,p) represents the situation that f(v) = p in the domination f that we are aiming to compute . We define the weight of each vertex (v,p) to be p.

The role of u is to define the "left" endpoint of the path that we will compute. All edges will go from "left" to "right". We partition the vertex set G_u into four subsets:

- $A_u = \{(v,p) \mid G(V \setminus B(v,p)) \text{ is connected and } u \in B(v,p)\}$
- $B_u = \{(v,p) \mid G(V \setminus B(v,p)) \text{ has two connected components}\}$
- $C_u = \{(v,p) \mid G(V \setminus B(v,p)) \text{ is connected and } u \notin B(v,p)\}$
- $D_u = \{(v,p) \mid B(v,p) = V\}$

For each vertex (v,p) , let $L_u(v,p)$ be the connected components of $G(V \setminus B(v,p))$ that contains u (i.e. , the component to the left of B(v,p)) , and let $R_u(v,p)$ be the connected component of

$G(V \setminus B(v,p))$ that does not contain u (i.e. the component to the right of B(v,p)). Thus $L_u(v,p) = \phi$ for every $(v,p) \in A_u \cup D_u$, and $R_u(v,p) = \phi$ for every $(v,p) \in C_u \cup D_u$.

The edges of G_u are directed and defined as follows: a directed edge $(v,p) \rightarrow (w,q)$ is an edge of G_u if and only if all of the following three conditions are satisfied:

- $B(v,p) \cap B(w,q) = \phi \text{ in } G$
- $R_u(v,p) \neq \phi \text{ and } L_u(w,q) \neq \phi$
- $(N_G(B(w,q) \cap L_u(w,q)) \subset B(v,p)$ $\text{and } (N_G(B(v,p)) \cap R_u(v,p)) \subset B(w,q) \text{ in } G$

To restate the last requirement in plain text : B(v,p) must contain all neighbors of B(w,q) in $L_u(w,q)$, and B(w,q) must contain all neighbors of B(v,p) in $R_u(v,p)$.

Lemma 3. Given G = (V,E) and a vertex u in G, let $(v_1,p_1) \rightarrow (v_2,p_2) \rightarrow ...., \rightarrow (v_k,p_k)$ be a directed path in G_u with $(v_1,p_1) \in A_u \cup D_u$ and $(v_k,p_k) \in C_u \cup D_u$. Then for $1 \le i \le k$, the following is true: $\bigcup_{j=1}^{i-1} B(v_j,p_j) = L_u(v_i,p_i)$

$\text{and } \bigcup_{j=i+1}^{k} B(v_j,p_j) = R_u(v_i,p_i)$.

Proof. Observing that k = 1 if and only if the path contains a vertex of D_u , in which case the lemma follows trivially. Let us for the rest the proof assume that $k \ge 2$.

We first show that $\bigcup_{j=1}^{i-1} B(v_j,p_j) = L_u(v_i,p_i)$ by induction on i, starting from i = 1 and containing i = k.

Let us consider the base cases i=1 and i=2. When i=1 we must show that $L_u(v_1,p_1) = \phi$, which follows trivially since $(v_1,p_1) \in A_u \cup D_u$. When i =2, we need to show that. Since $(v_1,p_1) \rightarrow (v_2,p_2)$ is an edge of G_u and $L_u(v_1,p_1) = \phi$, we know that $N_G(B(v_1,p_1)) \subset B(v_2,p_2)$. By the definition of an edge of G_u, we also know that $N_G(B(v_2,p_2)) \cap L_u(v_2,p_2) \subset B(v_1,p_1)$. Thus there cannot exist a path between a vertex of $B(v_2,p_2)$ and a vertex of $B(v_1,p_1)$ and the result follows since $L_u(v_2,p_2)$ is not connected. For the induction step, assume that $\bigcup_{j=1}^{i-1} B(v_j,p_j) = L_u(v_i,p_i)$, and we will show that $\bigcup_{j=1}^{i} B(v_j,p_j) = L_u(v_{i+1},p_{i+1})$. Because of the edge $(v_i,p_i) \rightarrow (v_{i+1},p_{i+1})$, by the proof of Observation 1, we know that

$B(v_i,p_i) \subseteq L_u(v_{i+1},p_{i+1}) \text{ and } B(v_{i+1},p_{i+1}) \subseteq R_u(v_i,p_i)$

.Thus, by the induction assumption, $B(v_{i+1},p_{i+1})$ does not intersect with $\bigcup_{j=1}^{i} B(v_j,p_j)$. Again by the induction assumption $\bigcup_{j=1}^{i} B(v_j,p_j)$ is connected and contains u. As a consequence, we can conclude that $\bigcup_{j=1}^{i} B(v_j,p_j) \subseteq L_u(v_{i+1},p_{i+1})$. Now if $L_u(v_{i+1},p_{i+1})$ contains a vertex x that does not belong to $\bigcup_{j=1}^{i} B(v_j,p_j)$ then due to the induction assumption,

Published By:
Blue Eyes Intelligence Engineering
& Sciences Publication

there must be a path (possibly a single edge) between x and a vertex of $B(v_i, p_i)$ whose vertices are all outside of $\bigcup_{j=1}^{i} B(v_j, p_j)$. Consequently, $B(v_i, p_i)$ must have a neighbor y in $R_u(v_i, p_i)$ such that $x \notin B(v_{i+1}, p_{i+1})$, which contradicts the existence of the edge (v_i, pi) $\rightarrow$ (v_{i+1}, p_{i+1}).Thus $U_{j=1}^{i} B(v_j, p_j) = L_u(v_{i+1}, p_{i+1})$ and the proof of this part is complete.
Showing that $U_{j=i+1}^{k} B(v_j, p_j) = R_u(v_i, p_i)$ for $1 \le i \le k$ is completely analogous, and we skip this part.

Algorithm: Minimum Path Domination-MPD
Input: A graph G = (V,E).
Output: An efficient broadcast domination function f of minimum cost on G, such that G_f is a path

Begin
  for each vertex v in G **do**
    f(v) = 0;
  Let P be a dummy path with W(P) = rad(G)+1;
For each vertex u in G **do**
    Compute G_u with vertex sets A_u, B_u, C_u *and* D_u;
    Find a minimum weight path P_u starting in vertex of $A_u \cup D_u$ and
    ending in a vertex of $C_u \cup D_u$;
    if $W(P_u) < W(P)$ then
      $P = P_u$;
  end-for
  for each vertex (v,p) on P do
    f(v) = p;
end.

IV. OPTIMAL SCATTERED DOMINATION FOR ALL CASES

Now we want to compute an optimal broadcast domination for any given graph G.Our approach will be as follows. Let x be any vertex of G. for each k between 1 and rad(G) such that $G' = G\{V \setminus B(x,k)\}$ is connected or empty , we run the minimum path broadcast domination algorithm MPD on G'. Our algorithm for the general case is given in figure 3.
In this way, we consider all domination f whose corresponding domination graphs are paths or cycles. The advantage of this approach is its simplicity. The disadvantage is that we also consider many cases that do not correspond to a path or a cycle, which we could have detected with a longer and more involve algorithm. However, these unnecessary cases do not decrease the asymptotic time bound.
Theorem 3. Algorithm OBD computes an optimal domination of any given graph.
Proof. Let G = (V,E) be the input graph . By theorem 1 and corollary 2, there is a vertex x in V and an integer $k \in [1, rad(G)]$ such that the graph $G' = G\{V \setminus B(x,k)\}$ has an efficient optimal scattered domination f' where the domination graph $G'_{f'}$ is a path , and that f' can extended to an optimal domination f for G with f(x) = k, f(v) = 0 for $v \in B(x,k)$ *with* $x \ne v$ *and* $f(v) = f'(v)$ for all other vertices v. Algorithm MPD computes an optimal domination of G',and since Algorithm OBD tries all possibilities for (x ,k), the result follows.

Algorithm: Optimal Broadcast Domination-OBD
Input: A graph G = (V, E).
Output: An efficient optimal broadcast domination function f on G.
begin
  opt = rad (G) +1;
  for each vertex x in G do
    for k = 1 to rad(G) do
      if $G' = G\{V \setminus B(x,k)\}$ is connected or empty then
        f = MPD(G');
        if $c_f(V \setminus B(x,k)) + k < opt$ *then*
          opt = $c_f(V \setminus B(x,k)) + k$;
        f(x) = k ;
        for each vertex v in B(x , k) \{x}do
    end-if
  end-if
end.

Note that although there is always an efficient optimal domination f such that G_f is a cycle or a path, there can of course exist other optimal dominations f' *with* $c_{f'}(V) = c_f(V)$ such that $G_{f'}$ is not a path or a cycle, and such that f' is not efficient. The optimal broadcast domination returned by algorithm OBD does not necessarily correspond to a path or a cycle.
e full paper, due to limited space here.

V. CONCLUSION

In this paper we have shown that the broadcast domination problem is solvable in polynomial time on all graphs. Our focus has been on polynomial time and not the best possible time bound. Our algorithm can be enhanced to run substantially faster, as explained. For further research, more efficient algorithms for this problem should be of interest.

REFERENCES:

1. J. Bar, G. Kortsarz and D.Peleg. How to allocate network centers. J. Algorithms 15,385-415, (1997).
2. J.R.S. Blair, SHorton. Broadcast domination algorithms for interval graphs and trees. J.Graph Theory, 169, 55-77(1999).
3. J.E.Dunbar, T.W. Haynes; Broadcast in graphs .J. Graph Theory, 30, 99-107(2000).
4. D.J. Ewin. Dominating broadcasts in graphs. J. Comp. Sci.April, 42, 89-105(2001).
5. M.R. Garey and D.S.Johnson, Computer and intra stability, J.Comp. Sci., 23, 446-450(2002).
6. T.W. Haynes, Domination in graphs J. Combinatory, 9, 23-29, (2003).

7. T.W. Haynes, S.T. Hedetniemi, Fundamentals of domination in Graphs , J. Graph theory , Vol.3, 52-59, (2004).
8. M.A. Henning, Distance domination in graphs, J. Combinatory, 321-349, (2005).
9. S.B. Horton, A. Mukherjee; a Computational study of the broadcast domination problem, J. Of Discrete Maths, 131-147, (2005).
10. P.J. Slater, R-domination in graphs, J. Computer Science, 23, 446-450, (2006).
11. M. Yannakaki's , F. Gavril, "Edge dominating sets in graphs, SIAM J. Applied Maths, 38, 364-372, (2007).