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Wavelets for the Fast Solution of Ordinary Differential Equations

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Abstract*— In this paper, wavelets have shown to be a powerful tool and a potential substitute for the Fourier transform in many problems. It is natural to use them for the solution of differential equations. In this chapter, we show how to use wavelets in the numerical solution of boundary value ordinary differential equations. Rather than using algebraic wavelets, we adapt the wavelets to the specific operator at hand. We want their construction to be easy to implement and computationally inexpensive in order to build a general solver

Index Terms: FFT, Wavelet, Boundary Value

Existing methods

Consider a linear ordinary differential equation of the form

$$Lu(x) = f(x) \quad for\ x \in [0,1],\ where\ L = \sum_{j=0}^{m} a_j(x) D^j,$$

with the boundary conditions

$$(B_0 u)(0) = g_0 \quad and \quad (B_1 u)(1) = g_1,$$

where

$$B_i = \sum_{j=0}^{m} b_{i,j} D_j \cdot$$

Presently, two major numerical solution techniques exist. First, if the coeffi- cients $a_j(x)$ are independent of *x*, the Fourier transform is best suited for solving the equation. The underlying reason is that the complex exponentials are eigenfunctions of a constant coefficient operator. In the Fourier basis, the operator becomes diagonal and can thus trivially be inverted. The algorithm consists of calculating the Fourier transform of the right-hand side, dividing the coefficient of each basis function by the corresponding eigenvalue and taking the inverse Fourier transform. This can be implemented using the FFT with a complexity of *O(MlogM)*, where ***M*** is the number of discretization points.

If the coefficients $a_j(x)$ are not constant, one typically uses finite element or finite difference methods. We focus here on the former and define the *operator inner product* associated with a self-adjoint operator L as

$$\langle\langle u, v \rangle\rangle = \langle Lu, v \rangle$$

In a Galerkin method, one considers two spaces $S\ and\ S^*$ and looks for an approximate solution $\bar{u} \in S$ so that.

$$\forall v \in S^* : \langle\langle \bar{u}, v \rangle\rangle = \langle f, v \rangle.$$

If $S\ and\ S^*$ are finite dimensional spaces with the same dimension, this leads to a linear system of equations. The matrix of this system is called the stiffness matrix. Its elements are the operator inner products of the basis functions of $S\ and\ S^*$.

Traditionally, one uses local finite elements which lead to a banded stiffness matrix. Since the matrix is sparse, the linear system is traditionally solved with an iterative method. Local finite elements, however, have the disadvantage that the stiffness matrix becomes ill-conditioned as the problem size grows. Typically, its condition number grows as a power of the number of elements. This slows down the convergence rate of the iterative algorithm dramatically.

This problem can be solved by using multiresolution techniques such as multigrid or hierarchical basis functions. Multiresolution finite element bases can provide preconditioners that result in a bounded condition number. The convergence rate of the iterative solver is then independent of the problem size.

One possible way of using wavelets is to take (bi)orthogonal algebraic wavelets as basis functions in a Galerkin method. This was proposed by several researchers. It results in a linear system that is sparse because of the compact support of the wavelets, and that, after preconditioning, has a condition number independent of problem size because of the multiresolution structure. However, in this setting the wavelets do not provide significantly better results than the other multiresolution techniques. In fact, one of their major properties, namely their (bi)orthogonality, is not fully exploited.

Three questions are addressed in this chapter:

1. How can one make use of the (bi)-orthogonality property of the wavelets?
2. Can wavelets diagonalize di_erential operators?
3. Can one construct fast algorithms?

General idea

We assume that L is self-adjoint and positive definite. Now write $L = \upsilon^* \upsilon$, where υ^* is the adjoint of υ. We call υ the *square root operator* of L. Note that it is not unique.

Suppose that the functions $\{\Psi_{j,k}\}$ and $\{\Psi^*_{j,k}\}$, for an appropriate range of indices, are bases for $S\ and\ S^*$ respectively. The entries of the stiffness matrix are then given by

$$\langle\langle \Psi_{j,k}, \Psi^*_{j',k'} \rangle\rangle = \langle L\Psi_{j,k}, \Psi^*_{j',k'} \rangle = \langle \upsilon\Psi_{j,k}, \upsilon\Psi^*_{j',k'} \rangle.$$

The idea is to let

$$\Psi_{j,k} = \upsilon^{-1}\Psi_{j,k} \ and\ \Psi^*_{j,k} = \upsilon^{-1}\tilde{\psi}_{j,k}$$

Where $\psi_{j,k}$ and $\tilde{\psi}_{j,k}$ are biorthogonal wavelets. Because of the biorthogonality, the sti_ness matrix becomes a diagonal matrix and thus can trivially be inverted. This avoids the use of an iterative algorithm. We call the $\Psi_{j,k}$ and $\Psi^*_{j,k}$ functions the operator wavelets and the $\psi_{j,k}$ and $\tilde{\psi}_{j,k}$ functions the original wavelets. The operator wavelets are

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biorthogonal with respect to the operator inner product, a property we refer to as operator biorthogonality. When using the operator wavelets as basis functions in a Galerkin method, the stiffness matrix becomes diagonal. Note that one can obtain diagonalization even though the operator wavelets are not eigenfunctions. This is not always a "true" diagonalization, as the operator wavelets and dual operator wavelets can differ. However, computationally the difference is irrelevant. This idea has potential provided one can find a fast numerical algorithm to compute the operator wavelets. Therefore, the operator wavelets need to generate an operator multiresolution analysis with an associated fast wavelet transform. This implies compactly supported operator wavelets and operator scaling functions $\Phi_{j,k}$. We will see that the latter cannot be constructed by simply applying υ^{-1} to the original scaling functions.

The analysis is relatively straightforward for simple constant coefficient operators such as the Laplace and polyharmonic operator. The reason is that these operators preserve the algebraic structure of wavelets. The construction of the operator wavelets can thus rely on the Fourier transform. The situation becomes different for more general constant coe_cient and for variable coe_cient operators. We show how one then can use weighted wavelets.

The idea to adapt wavelets to a differential operator is also suggested elsewhere. ***Stefan Dahlke*** and ***Ilona Weinreich*** construct wavelets that are operator semiorthogonal. As a result, one does not obtain a full diagonalization, but rather a decoupling of equations corresponding to different levels. Antiderivates of wavelets are used in a Galerkin method. This parallels our construction in the case of the Laplace or polyharmonic operator. Our idea also is different from the technique Gregory Beylkin presents in [22]. He uses algebraic wavelets for the rapid calculation of the inverse of the matrix coming from a finite difference discretization. He also shows that the wavelets provide a diagonal preconditioner that yields uniformly bounded condition numbers.

Harmonic operators

The one-dimensional Laplace operator and a possible square root are

$$L = -D^2 \quad and \quad \upsilon = D.$$

The associated operator inner product is therefore the homogeneous Sobolev innerproduct,

$$\langle\langle u, v\rangle\rangle = \langle u', v'\rangle.$$

Since the action of υ^{-1} is simply taking the antiderivative, the operator motherwavelets are given by

$$\Psi(x) \int_{-\infty}^{x} \psi(t)\, dt, \quad and \quad \Psi^*(x) \int_{-\infty}^{x} \tilde{\psi}(t)\, dt.$$

Here ψ and $\tilde{\psi}$ are compactly supported biorthogonal algebraic wavelets. The operator wavelets are also compactly supported because the integral of the original wavelets vanishes. Since translation and dilation is preserved, we define the operator wavelets as

$$\Psi_{j,k}(x) = \Psi(2^{-j}x - k) \quad and \quad \Psi^*_{j,k}(x) = \Psi^*(2^{-j}x - k)$$

It immediately follows that

$$\langle\langle \Psi^*_{j,k}, \Psi_{j',k'} \rangle\rangle = 2^j\, \delta_{j-j'} \delta_{k-k'}.$$

Consequently, the stiffness matrix is diagonal with powers of 2 on its diagonal.

We now define the spaces

$$W_j = span\{\Psi_{j,k} : k \in \mathbb{Z}\}.$$

We want to find the associated multiresolution analysis. In other words, we needspaces V_j so that

$$V_{j+1} = V_j \oplus W_j$$

and operator scaling functions $\Phi_{j,k}$ so that

$$V_j = span\{\Phi_{j,k} : k \in \mathbb{Z}\}.$$

These spaces are closed as they are finite dimensional.

The antiderivative of the original scaling function is not compactly supportedand hence cannot be used as an operator scaling function. We instead construct the operator scaling function Φ by taking the convolution of the original scalingfunction with the indicator function on [0, 1],

$$\Phi = \phi * \chi_{[0,1]},$$

and let

$$\Phi_{j,k}(x) = \Phi(2^j x - k).$$

Note that

$$\Phi'(x) = \phi(x) - \phi(x+1).$$

We next show that the V_j spaces are nested and that W_j complements V_j in V_{j+1}.

In the Fourier domain we have

$$\hat{\Phi}(\omega) = \frac{1 - e^{-i\omega}}{i\omega}\, \hat{\phi}(\omega),$$

and

$$\hat{\Psi}(\omega) = \frac{1}{i\omega}\, \hat{\psi}(\omega).$$

A simple calculation shows that the operator scaling function satisfies a refinement equation.

$$\hat{\Phi}(\omega) = \hat{\Phi}\left(\frac{\omega}{2}\right) H\left(\frac{\omega}{2}\right) \quad with \quad H(\omega) = \frac{1 + e^{-i\omega}}{2}\, h(\omega).$$

Consequently, the V_j spaces are nested. The space W_j is a subset of V_{j+1} if a trigonometric polynomial G exists so that

$$\hat{\Psi}(\omega) = \hat{\Phi}(\omega/2) G(\omega/2).$$

Substituting this in (4.1) yields,

$$G(\omega/2)\, \hat{\Phi}(\omega/2) = \frac{1}{i\omega}\, \mathrm{g}(\omega/2) \hat{\phi}(\omega/2)$$

It then follows that

$$G(\omega) \hat{\phi}(\omega) \frac{1 - e^{-i\omega}}{i\omega} = \frac{1}{2i\omega}\, \mathrm{g}(\omega)\, \hat{\phi}(\omega)$$

$$G(\omega)(1 - e^{-i\omega}) = \mathrm{g}(\omega)/2$$

$$G(\omega) = \frac{1/2}{1 - e^{-i\omega}}\, \mathrm{g}(\omega).$$

This function is a trigonometric polynomial, because g is a trigonometric polynomialwith g (0) = 0.

The space W_j complements V_j in V_{j+1} if

$$\Delta(\omega) = \det\begin{bmatrix} H(\omega) & H(\omega+\pi) \\ G(\omega) & G(\omega+\pi) \end{bmatrix}$$

does not vanish, see Section 2.5. We readily see that

$$\Delta(\omega) = \delta(\omega)/4,$$

and $\delta(\omega)$ doesn't vanish since ϕ and ψ generate a multiresolution analysis. The construction of the dual functions Φ^* and Ψ^* from $\tilde{\phi}$ and $\tilde{\psi}$ is completely similar. The coefficients of the trigonometric polynomials H, H^*, G and G^* now define the fast wavelet transform associated with the operator inner product.

Algorithm

We describe the algorithm in the case of periodic boundary conditions. This implies that the basis functions on the interval [0, 1] are the periodization of the basis functions on the real line,

Let $S = V_n$ (respectively $S^* = V_n^*$) and consider a basis $\{\Phi_{n,k} : 0 \le k < 2^n\}$ (respectively $\Phi^*_{n.k}$). Let $M = 2^n$. Define a vector $b \in C^M$ as

$$b_k = \langle f, \Phi^*_{n.k} \rangle \quad with \ 0 \le k < M,$$

and a vector $x \in C^M$ so that we can write $\bar{u} \in S$ as

$$\bar{u} = \sum_{k=0}^{M-1} x_k \ \Phi_{n,k}.$$

The Galerkin method with these bases then yields a linear system $Ax = b \ \ with \ \ A_{k,l} = \langle\langle \Phi^*_{n,l}, \Phi_{n,k} \rangle\rangle$.

As we mentioned earlier, the matrix A cannot be diagonal. Also, its condition number grows as $O(M^2)$. Consider the decomposition

$$V_n = V_0 \oplus W_0 \oplus \ldots\ldots \oplus W_{n-1},$$

and the corresponding wavelet basis. The space V_0 has dimension one and contains constant functions. We switch to a one index notation so that the sets.

$$\{1, \Psi_{j,k} : 0 \le j < n, 0 \le k < 2^j\} \ and \ \{\Psi_k : 0 \le k < 2^n\}$$

Coincide and similarly for the dual functions. Define the vectors b' and x' of C^M so that

$$b'_k = \langle f, \Psi^*_k \rangle \ \ and \ \ \bar{u} = \sum_{k=0}^{M-1} x'_k \ \Psi_k.$$

We know that matrices T and T^* exists so that

$$b' = T^* b \ \ and \ \ x = T x'.$$

The action of the matrix T (respectively T^*) can be implemented using the fast wavelet transform decomposition with filters H and G (respectively H^* and G^*). The complexity of the matrix vector multiplication is O(M). In the wavelet basis the system becomes

$$A'x' = b' \ \ with \ A' = T^* AT,$$

where $A'_{k,l} = \langle\langle \Psi^*_{n,l}, \Psi_{n,k} \rangle\rangle$.

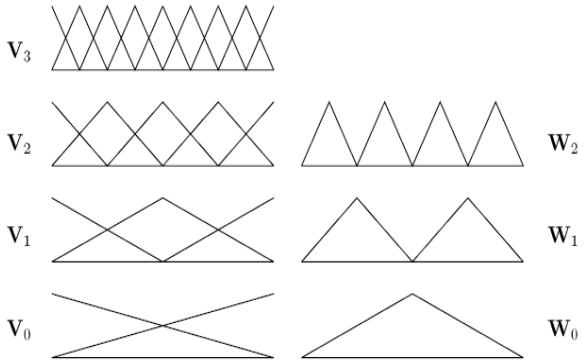

Figure : Basis for Dirichlet problem.

Since A' is diagonal, it can be trivially inverted. The coefficients of the solution in the scaling function basis are given by

$$x = TA'^{-1} T^* b.$$

The algorithm consists of calculating the wavelet decomposition of the right-hand side, dividing each coefficient by its corresponding diagonal element and reconstructing to find the solution. The complexity is *O(M)*.

The constant basis function of V_0 has a zero as corresponding diagonal element and its coefficient is thus undetermined. Indeed, the solution is only defined up to a constant. This does not lead to a division by zero as the integral of *f* has to vanish,

$$\int_0^1 f(x)\, dx = u'(1) - u'(0) = 0.$$

In the next section we will discuss how to deal with other boundary conditions.

Example

In this section we take a look at a simple example, namely the basis we get starting from the Haar wavelets. Remember that

$$\phi(x) = \chi_{[0,1]}(x) \ \ and \ \ \psi(x) = \phi(2x) - \phi(2x-1).$$

It immediately follows that both the operator wavelet and scaling functions are B-splines of order 2 (hat functions),

$$\Phi(x) = \Lambda(x) \ \ and \ \ \Psi(x) = \Lambda(2x).$$

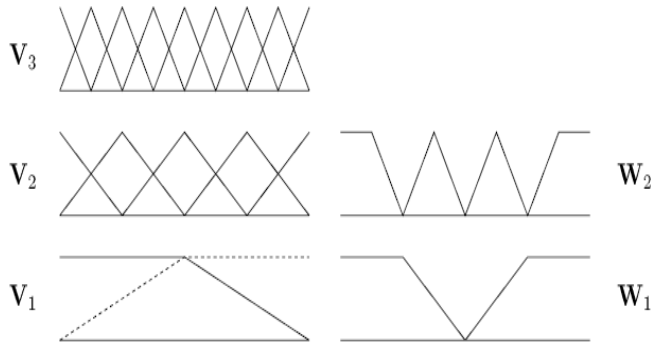

Figure : Basis for Neumann problem.

The original wavelets are orthogonal and as a consequence the basis functions and dual functions coincide.

The operator scaling functions satisfy the Strang-Fix condition with $N = 2$ and the convergence thus is of order h^2. One can prove that higher order wavelets with more vanishing moments (*N*) will, in general, not yield faster convergence because the solution u is not smooth enough. The underlying

reason is that iff $f \in L_2([0,1])$, the solution u belongs to the Sobolev space $H_{0,0}([0,1])$. One can get faster convergence only by imposing extra regularity conditions on the right-hand side. In a way this basis seems to be the most natural one to work with. Note that these piecewise linear basis functions are local solutions of the homogeneous equation. Hence the operator scaling functions and wavelets are υ-splines. This basis also coincides with Yserentant's hierarchical basis.

The idea to deal with boundary conditions is to let the operator wavelets satisfy the homogeneous boundary conditions and to let the component in the V_0 space satisfy the imposed boundary conditions. Figure 4.1 shows the basis functions in the case of Dirichlet boundary conditions and n = 3. All operator wavelets vanish at the boundary. The coefficients of the two functions in the V_0 space are determined by the boundary conditions. The fast wavelet transform differs from the periodic algorithm in the sense that different filter coefficients are used for the wavelets at the boundary. Note the "half hat" functions at the boundary.

The basis in case of the Neumann problem is shown in Figure. All operator wavelets have derivative zero at the boundary. The boundary conditions are handled by the two functions in the V_1 space. Again the coefficient of the constant is undetermined. The integral of *f* is now equal to $u'(1) - u'(0)$.

The polyharmonic operator

The polyharmonic equation is defined as $-D^{2m} u = f$ and we take the square root operator to be

$$\upsilon = D^m$$

The operator scaling function Φ is now *m* times the convolution of the original scaling function ϕ with the box function, and the operator wavelet Ψ is *m* times the antiderivative of the original wavelet ψ. In order to get a compactly supported wavelet, the original wavelet needs to have at least *m* vanishing moments, a property that can be satisfied by all known wavelet families. The construction and algorithm are then completely similar to the case of the Laplace operator.

The Helmholtz operator

The one-dimensional Helmholtz operator is given by

$$L = -D^2 + k^2$$

so that we can take

$$\upsilon = D + k$$

We assume that *k = 1*, which can always be obtained from a linear transformation. Observe that $\upsilon = D + I = e^{-x} D\, e^{x}$ and thus $\upsilon^{-1} = e^{-x} D^{-1}\, e^{x}$

Applying υ^{-1} to a wavelet does not necessarily yield a compactly supported function, since $e^{x}\psi_{j,k}$ does not have a vanishing integral. Therefore, we let $\Psi_{j,k} = \upsilon^{-1} e^{-x} \psi_{j,k} = e^{-x} D^{-1} \psi_{j,k}$.

If $\psi_{j,k}$ has a vanishing integral, then $\Psi_{j,k}$ is compactly supported. In order to diagonalize the stiffness matrix, the original wavelets now need to be orthogonal with respect to a weighted inner product with weight function e^{-2x} because

$$\begin{aligned} \left\langle\left\langle \Psi_{j,k},\, \Psi^{*}_{j',k'} \right\rangle\right\rangle &= \left\langle \upsilon \Psi_{j,k}, \upsilon \Psi^{*}_{j',k'} \right\rangle \\ &= \left\langle e^{-x} \psi_{j,k}, e^{-x} \tilde{\psi}_{j',k'} \right\rangle \\ &= \int_{-\infty}^{+\infty} e^{-2x} \psi_{j,k}(x) \tilde{\psi}_{j',k'}(x)\, dx. \end{aligned}$$

We see that the original wavelets need to be weighted wavelets. In this section we only use the unbalanced Haar wavelets as weighted wavelets. Theorthogonality of the unbalanced Haar wavelets on each level immediately follows from their disjoint support, since $supp\ \psi_{j,k} = \left[2^{-j}k,\ 2^{-j}(k+1)\right]$. To get orthogonality between the different levels, V_j has to be orthogonal to $W_{j'}$ for $j' \geq j$ or

$$\int_{-\infty}^{+\infty} e^{-2x} \phi_{j,k}(x)\, \tilde{\psi}_{j',k'}(x)\, dx \quad for \quad j' \geq j.$$

We let the scaling function coincide with e^{2x} on the support of the finer scale wavelets,

$$\phi_{j,k} = e^{2x}\ \chi_{j,k},$$

Where $\chi_{j,k}$ is the indicator function on the interval $\left[2^{-j}l,\ 2^{-j}(l+1)\right]$, normalized so that the integral of the scaling functions is a constant. We choose the wavelets as

$\psi_{j,k} = \phi_{j+1,2k} - \phi_{j+1,2k+1}$, So that they have a vanishing integral. The orthogonality between levels now follows from the fact that the scaling functions coincide with e^{2x} on the supportof the finer scale wavelets, and from the vanishing integral of the wavelets

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{-2x} \phi_{j,k}(x) \tilde{\psi}_{j',k'}(x)\, dx &= \int_{-\infty}^{+\infty} \chi_{j,k}\, \tilde{\psi}_{j',k'}(x)\, dx \\ &= C \int_{-\infty}^{+\infty} \tilde{\psi}_{j',k'}(x)\, dx = 0 \end{aligned}$$

One can see that the operator wavelets are now piecewise hyperbolic functions (piecewise combinations of e^{x} and e^{-x}). The operator scaling functions are chosen as

$$\Phi_{j,k} = e^{-x} D^{-1}\left(\phi_{j,k} - \phi_{j,k+1}\right),$$

So that

$$\Psi_{j,k} = \Phi_{j+1,2k}.$$

With the right normalization, one gets

$$\Phi_{j,k}(x) = \begin{cases} \dfrac{\sinh\left(x - k2^{-j}\right)}{\sinh\left(2^{-j}\right)} & for\ x \in \left[k2^{-j}, (k+1)2^{-j}\right] \\ \dfrac{\sinh\left((k+1)2^{-j} - x\right)}{\sinh\left(2^{-j}\right)} & for\ x \in \left[(k+1)2^{-j}, (k+2)2^{-j}\right] \\ 0 & elsewhere \end{cases}$$

The operator scaling functions on one level are translates of each other but the ones on different levels are no longer dilates of each other. They are supported on exactly the same sets as the ones in Figure and they roughly look similar. The operator scaling functions satisfy a refinement relation

$$\Phi_{j,k} = \sum_{l=0}^{2} H_{j,l} \Phi_{j+1,2k+l},$$

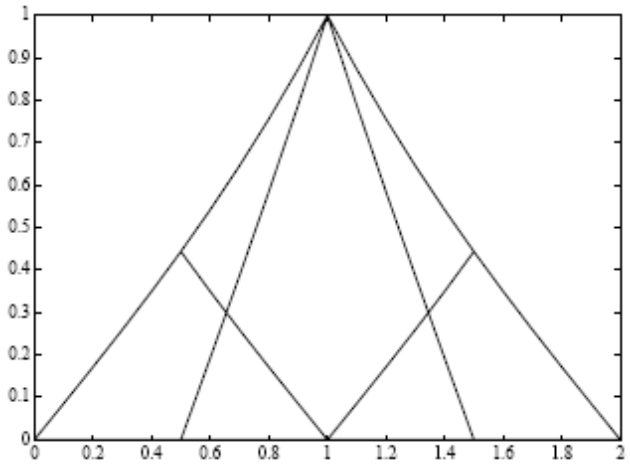

Figure: The refinement relation for the operator scaling functions.

with

$$H_{j,0} = H_{j,2} = \sinh\left(2^{-j-1}\right)/\sinh\left(2^{-j}\right) \quad and \quad H_{j,1} = 1.$$

The three scaling functions on the finer scale are not the dilates of the one on the coarse scale, but they still add up to it.

The Helmholtz operator in the basis of operator wavelets again is diagonal and the algorithm is completely similar to the Laplace case. The only difference in implementation is that the filters in the fast wavelet transform now depend on the level.

Note that these functions again are υ -splines and, in a way, are the most natural to work with. Also note that

$$\lim_{j\to\infty} \Phi_{j,0}\left(2^{-j}x\right) = \Lambda(x).$$

Despite the fact that the Strang-Fix conditions are not satisfied, one can prove that the convergence is still of order h^2.

We conclude that a wavelet transform can diagonalize constant coefficient operators similarly to the Fourier transform. The resulting algorithm is faster (*O(M)* instead of ***O(M logM)*).** This gain in speed is a consequence of the subsampling on the coarser levels in the wavelet transform (the ones that correspond to the low frequency components of the solution), which is not present in the Fourier transform.

CONCLUSION

In this paper we showed how wavelets can be adapted to be useful in the solution of differential equations. Like the Fourier transform, wavelets can diagonalize constant coefficient operators. The resulting algorithm is slightly faster. The main result, however, is that even non-constant coefficient operators can be diagonalized with the right choice of basis. This evidently yields a much faster algorithm than the classic iterative methods.

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