Exact Solutions of Time Dependent Korteweg-de Vries Equation by the Extended Unified Method

H. I. Abdel-Gawad, Mohamed Osman, Nasser S. Elazab

Abstract: ***— Recently the unified method for finding traveling wave solutions of nonlinear evolution equations was proposed by the first author. It was shown that, this method unifies all the methods being used to find these solutions. In this paper, we extend this method to find a class of formal exact solutions to Korteweg-de Vries equation with time dependent coefficients. A new class of multiple-soliton or wave trains is obtained.***

Index Terms— Exact solution, Extended unified method, Korteweg-de Vries equation, Variable coefficients

I. INTRODUCTION

We consider the following evolution equation

$$f\left(x,t,u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x\partial t},\frac{\partial^2 u}{\partial x^2},...,\frac{\partial^m u}{\partial x^m}\right)=0,\quad m\geq 1, \tag{1}$$

where f is a polynomial in its arguments. When Eq. (1) does not depend explicitly on x and t, it can be reduced to a subclass of ordinary differential equations by using the Lie groups for partial differential equations [1] or by using similarity transformations. Among these equations, the traveling wave has the form

$$g(u,u',u'',...u^{(m)})=0,\ u'=\frac{du}{dz},\ z=x-ct, \tag{2}$$

which results due to the translation symmetry of (1). The Painleve' analysis is used to testing the integrability of partial differential equations, that was developed in [2]. $Auto-B\ddot{a}cklund$ transformation deals with the exact solutions that were obtained for integrable forms of (2) by truncating Painleve' expansion [3-9].

Recently $auto-B\ddot{a}cklund$ transformation that was extrapolated in [10-14] and the homogeneous balance method in [15-19] assert a solution for evolution equations with variable coefficients in the form

$$u(x,t)=\frac{\partial^{m-2}}{\partial x^{m-2}}(a(\phi)\phi_x)+u^{(0)}(x,t),$$

where ϕ is the base function.

II. EXTENDED UNIFIED METHOD

Explicit solutions of Eq. (2) are, in fact, particular solutions. In this respect, these solutions are mapped to other solutions that are given in terms of known elementary or special functions. Recently in [20] the class of these solutions were obtained by the generalized mapping method (GMM). This method generalizes the results as a polynomial or a rational function solutions. In the present paper, we extend this method to handle equations of type (1).

A. *Polynomial solutions*

In this section, we search for polynomial solutions of Eq. (2) in $C^s(\mathrm{R})$ (*the class of continuously partially differentiable functions up to order s*), and we define the set of functions

$$S=\{\phi:\mathrm{R}\times\mathrm{R}^+\to\mathrm{K}\subset\mathrm{R},\ \phi_t^q=P_{k_1}^t(\phi),(\phi_x)^p=P_k^x(\phi)\},$$

$$P_{k_1}^t(\phi)=\sum_{i=0}^{k_1}b_i(x,t)\phi^i(x,t),P_k^x(\phi)=\sum_{i=0}^{k}c_i(x,t)\phi^i(x,t).$$

Indeed the set S contains elementary or elliptic functions for some particular values of q,p,k_1 and k. The mapping method asserts that there exists a positive integer n and a mapping

$$M:C^s(\mathrm{R})\to\Omega,\quad \Omega=\{v,v=\sum_{i=0}^{s}a_i(x,t)\varphi^i,\varphi\in S\},$$

such that $M(u)=P_n(\phi)$ and satisfies the properties

$$M(\alpha_1u_1+\alpha_2u_2)=\alpha_1M(u_1)+\alpha_2M(u_2),$$
$$M(u_1u_2)=M(u_1)M(u_2),\quad M(u_t)=(M(u))_t,$$
$$M(u_x)=(M(u))_x.$$

Thus M is a ring homomorphism that conserves differentiation. By the former conditions we find that,

$$M(u_t)=P_{(n-1+k_1)}^t(\phi)\in\Omega,\ M(u_x)=P_{(n-1+k)}^x(\phi)\in\Omega.$$

By using the properties of M and the last results and as $f\equiv f(x,t,u,u_t,...)$ is a polynomial in its arguments, we find that $M(f)$ is a polynomial and there exists $s_o\leq s$ such that $M(f)=P_{s_o}(\phi)\in\Omega$. It is worthy to notice that all these polynomials have different coefficients. More simply the mapping M assigns to u and f gives two auxiliary equations, the polynomials $P_n(\phi)$ and $P_{s_o}(\phi)$ respectively. In case of Eq. (1) $s_o=n-m+mk$. The utility of the above presentation helps us to give arguments to the statements of the conditions in lemmas 2.1 and 2.2. Also, we think that it allows for constructing more generalization and it is more appropriate when (1) is a vector equation.

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We substitute for $u, u_t, u_x, ..., \frac{\partial^m}{\partial x^m}u$ as polynomials in ϕ, so that the function f is a polynomial in ϕ, together with two auxiliary equations. In the applications we may write directly $u = P_n(\phi)$, and $f = P_{s_o}(\phi)$. From the previous analysis we may write

$$u = \sum_{i=0}^{n} a_i(x,t)\phi^i, \qquad (3)$$

where for instance we assume that $k_1 = k$, so that the auxiliary equations are

$$\phi_t = \sum_{i=0}^{k} b_i(x,t)\phi^i, \ \phi_x = \sum_{i=0}^{k} c_i(x,t)\phi^i, \qquad (4)$$

together with the compatibility equation

$$\phi_{xt} = \phi_{tx}. \qquad (5)$$

We mention that solutions of (4) when exist, are elementary ($p = q = 1$). The case of elliptic solutions ($p = q = 2$) will be considered in a future work.

When substituting from (3) and (4) into (1) we find that it is transformed to $P_{s_o}^{(f)}(\phi) \equiv 0$ that gives rise to

$$\sum_{i=0}^{s_o} h_i(a_{r_o}(x,t), b_{r_1}(x,t), c_{r_2}(x,t), a_{r_o t}(x,t), a_{r_o x}(x,t), ...)\phi^i \equiv 0, \qquad (6)$$
$$r_0 = 0,1,...,n \quad and \quad r_1, r_2 = 0,....,k.$$

By equating the coefficients of $\varphi^i, i = 0,1,..., s_o$ to zero, we get a set of $(s_o + 1)$ algebraic (or differential) equations, namely the principle equations, in the functions a_i, b_i and c_i. On the other hand the equations that result from (5) count: $2k - 1$, $k \geq 2$. We mention that these later unknown functions count: $n + 2k + 3$.

In Eq.(1), if $u^j u_x$ and $\frac{\partial^m}{\partial x^m}u$ are the highest nonlinear and the highest order derivative terms respectively, then we get the balancing condition as $s_o = nj + n + k - 1 = n - m + mk$. Thus by solving for n, we find that it depends on m, j and k. The last result and the number of compatibility equations namely $2k - 1$, $k \geq 2$ determine if the equations to be solved are over-determined or under-determined. The number of the determining equations, balances the number of unknowns, is over-determined or is under-determined when the difference,namely $(n - m + mk + 1) + (2k - 1) - (n + 2k + 3)$ is equal to 0, > 0, or < 0 respectively. From this last conditions, we may determine a consistency condition that will be identified in the lemmas. In what follows necessary conditions for the existence of polynomial solutions will be stated.

Lemma 2.1. *For polynomial-solutions of (1) (as a polynomial in* ϕ*) to exist it is necessary that*

(i) $(m-1)(k-1)/j (:= n)$ *is a positive integer*

(ii) $m(k-1) - 3 \leq m$ *when the equation (1) in the absence of* x*, and* t *passes the Painleve' test . Otherwise* m *is replaced by 2.*

We notice that the first and the second conditions in lemma 2.1 are the balancing and the consistency conditions respectively. For details see [20].

B. The rational function solutions

Here, also we search for solutions of Eq. (1) in $C^s(\mathbb{R})$. For rational function-solutions of Eq.(1), we consider the space of functions $\Omega_R = \{v, v = P_n(\phi)/Q_r(\phi), \phi \in S\}$ and $Q_r(\phi)$ has no zeros in $K \subset \mathbb{R}$. The definitions in the above and the GMM for rational function solutions assert that there exists a mapping

$$M_R : C^s(\mathbb{R}) \to \Omega_R, \quad M_R(u) = P_n(\varphi)/Q_r(\varphi), \ \varphi \in S.$$

The properties of these mapping are the same properties of the mapping $M(u)$ in section 2.1. By bearing in mind these properties and from (4), (5) we find that

$$M_R(u_t) = P_{1,(n-1+k+r)}(\varphi)/Q_r^2(\varphi),$$
$$M_R(u_x) = P_{2,(n-1+k+r)}(\varphi)/Q_r^2(\varphi),$$

thus $M_R(\frac{\partial^i u}{\partial x^i}) \in \Omega_R, i = 1,...,m$.

By using the properties of M_R and the last results, we get $M_R(f) \in \Omega_R$ and there exists $s_1 \leq s$ such that $M_R(f) = P_{s_1}(\varphi)/Q_r^{m+1}(\varphi)$. Indeed s_1 depends on n, r, k and also on m, where in the case mentioned in the above $s_1 = n - m + mk + mr$. Simply, we write

$$u = \sum_{i=0}^{n} a_i \phi^i / \sum_{i=0}^{r} d_i \phi^i. \qquad (7)$$

So that the Eq. (1) is transformed to $P_{s_1}(\varphi) \equiv 0$. Equivalently, the last identity becomes

$$\sum_{i=0}^{s_1} h_i(a_{r_0}(x,t), d_{r_2}(x,t), b_{r_1}(x,t), c_{r_3}(x,t), ...)\phi^i \equiv 0,$$
$$r_0 = 0,...,n, r_1, r_3 = 0,...,k, \quad and \quad r_2 = 0,...,r. \qquad (8)$$

In (8), by equating the coefficients of $\phi^i, i = 0,1,..., s_1$ to zero, we get a set of $(s_1 + 1)$ equations that determine the functions a_i, b_i, c_i and d_i. We mention that these later functions count $n + 2k + r + 3$. By using the same assumptions on Eq. (1), as in section 2.1, the balancing condition is

$$\begin{cases} nj + n + k - 1 + r = n + m\,(k + r - 1) + r\,(j - (m+1)), & m+1 < j \\ nj + r\,((m+1) - j) = n + m\,(k + r - 1) = s_1, & m+1 > j \end{cases} \qquad (9)$$

Now by solving (9) for n, we find that it depends on m, j, r and k and, in both two cases, we get the same equation for $n - r$. Hereafter, we distinguish between the two cases mentioned in (9). From the last results and when $j < m + 1$, the number of the determining equations, balances the number of unknowns, is over-determined or is under-determined when the difference, namely $(n-m+mk+rm+1)+(2k-1)-(n+2k+r+3)$ Is equal to $0, >0, or <0$ respectively.

But when $j>m+1$ this difference is $(n-m+mk+rm+1+r(j-(m+1)))+(2k-1)-(n+2k+r+3)$.

From these last conditions, we may determine the consistency condition that will be identified in the following Lemma.

Lemma 2.2. *For solitary wave-rational solutions of Eq. (2) to exist it is necessary that*

(i) $(m-1)(k-1)/j (:= n-r)$ *is an integer*

(ii) $r(m-1)+(k-1)m-3 \le m,\ j<m+1$

or $r(j-2)+(k-1)m-k-2 \le 2,\ j>m+1$, *in the case when Eq. (1) passes the Painleve'test. Otherwise* $r(m-1)+(k-1)m-k-2 \le 2,\ j<m+1$

or $r(j-2)+(k-1)m-k-2 \le 2,\ j>m+1$.

For details see [20].

III. EXACT SOLUTIONS OF TIME DEPENDENT KDV EQUATION

We consider the following KdV equation with variable coefficients

$$u_t + f(t)u_{xxx} + g(t)uu_x = 0, \qquad (10)$$

where f and g are arbitrary functions of t. We mention that (10) describes the propagation of waves in a medium with time-dependent dispersion and conviction.

In fact, Nirmala and Vedan [21] and E. Fan [12] studied (10) by using $auto-Bäcklund$ transformation. In this works, solutions of (10) were found in the case when $f(t)=cg(t)$, where c is a constant.

In this case Eq. (10) is transformed to the KdV equation with constant coefficients in the variables $\tau=\int g(t)dt$, $x=x$, and $u=u$. So that one gets the well known solutions for the KdV equation in the variable $x-c\tau$, namely soliton, solitary, or elliptic wave solutions. We mention that these solutions are bounded every where.

In this respect we have the following theorem.

Theorem 3.1. *There exists at least one bounded solution to the equation (10) if and only if* $f(t)=cg(t)$, *where* c *is a constant.*

Proof. When $f(t)=cg(t)$, from the previous arguments, it is a direct result that at least one bounded solution exists.

Now, we prove the converse. By using the contra-positive statement we assume that $f(t)\ne cg(t)$, then we show that there exists at most one unbounded solution (in the sense that $u(x,t)$ behaves asymptotically as x when $|x|\to\infty$). To this end we set $f(t)=c(t)g(t)$ and in the above new variables Eq. (10) becomes $u_\tau + c(\tau)u_{xxx} + uu_x = 0.$ Indeed this last equation has the particular solution $u=\frac{-12c(\tau)}{x^2}+\frac{x}{c(\tau)}$, $c(\tau)=A_0+\tau$, A_0 is a constant.

In the original variables,

$u=\frac{-12(A_0+\int g(t)dt)}{x^2}+\frac{x}{A_0+\int g(t)dt}$ which is unbounded. □

In what follows we shall find solutions of Eq. (10) when $f(t)\ne cg(t)$.

A. *The polynomial function solutions*

In lemma 2.1, the consistency condition holds when $k=2,3$ but it does not hold when $k\ge 4$. So that, only the cases $k=2,3$ will be considered.

- First case: When $k=2, n=2$, by substituting into (3), (4) and (10), we get six principle equations. We mention that calculations are carried out by using MATHEMATICA where standard functions in calculus and algebra were only needed. The steps of computations are as follows

Step 1. Solving the principle equations, where five of them are solved explicitly to

$$\begin{aligned}
&a_2(x,t)=-12h(t)c_2^2(x,t),\\
&a_1(x,t)=-12h(t)(c_1(x,t)c_2(x,t)+c_{2x}(x,t)),\\
&b_2(x,t)=\frac{-1}{c_2(x,t)k(t)}(a_0(x,t)c_2^2(x,t)+h(t)\\
&(c_1^2(x,t)c_2^2(x,t)+8c_2^3(x,t)c_0(x,t)+4c_2^2(x,t)\\
&c_{1x}(x,t)+2c_1(x,t)c_2(x,t)c_{2x}(x,t)-3c_{2x}^2(x,t)\\
&+4c_2(x,t)c_{2xx}(x,t)),
\end{aligned} \qquad (11)$$

and $h(t)=\frac{f(t)}{g(t)}$, $k(t)=\frac{1}{g(t)}$, $g(t)\ne 0, t\ge 0$.

The explicit equations for $b_1(x,t)$ and $b_0(x,t)$ are too lengthy to written here. It remains only one unsolved equation of the principle ones.

Step 2. We consider the compatibility equations that result from $\phi_{xt}=\phi_{tx}$ and they are given formally by;

$$\begin{aligned}
&b_0(x,t)c_1(x,t)-b_1(x,t)c_0(x,t)+c_{0t}(x,t)-b_{0x}(x,t)=0,\\
&2b_0(x,t)c_2(x,t)-2b_2(x,t)c_0(x,t)+c_{1t}(x,t)-b_{1x}(x,t)=0,\\
&-b_2(x,t)c_1(x,t)+b_1(x,t)c_2(x,t)+c_{2t}(x,t)-b_{2x}(x,t)=0.
\end{aligned} \qquad (12)$$

To simplify the computations, we make the transformations

$$c_{2x}(x,t)=p(x,t)c_2(x,t), c_1(x,t)=-p(x,t)+C_1(x,t), \qquad (13)$$

$$c_0(x,t)=\frac{-2C_{1x}(x,t)+C_1^2(x,t)+4C_0(x,t)}{4c_2(x,t)},$$

Where $C_0(x,t), C_1(x,t)$ are arbitrary functions.

By solving the equation in (12) for $a_0(x,t)$, we get

$$a_0(x,t)=a_{00}(t)-3C_1^2(x,t)h(t)-6C_0(x,t)h(t)+\frac{x\,k(t)h'(t)}{h(t)}, \qquad (14)$$

where $a_{00}(t)$ is an arbitrary function.

When substituting from (14) into (12) we find that the two equations $(12)_2$ and $(12)_3$ hold.

Step 3. Solving the reminded equations which are given by

$$H(h(t),k(t),a_{00}(t),C_0(x,t),...)=0,$$
$$h(t)k(t)a_{00}'(t)+a_{00}(t)k(t)h'(t)+x\,k(t)h'(t)k'(t)+ x\,k^2(t)h''(t)-6h(t)H(h(t),k(t),a_{00}(t),C_0(x,t),...)=0, \tag{15}$$
$$H(h(t),k(t),a_{00}(t),C_0(x,t),...)=x\,k(t)h'(t)C_{0x}(x,t)+ h(t)(k(t)C_{0t}(x,t)+a_{00}(t)C_{0x}(x,t))+2C_0(x,t)(k(t) h'(t)+3h^2(t)C_{0x}(x,t))+h^2(t)C_{0xxx}(x,t).$$

From (15) we should have

$$h(t)k(t)a_{00}'(t)+a_{00}(t)k(t)h'(t)+xk(t)h'(t)k'(t)+ xk^2(t)h''(t)=0, \tag{16}$$

which solves to

$$a_{00}(t)=\frac{A_1}{h(t)},\ h(t)=(h_1+h_0\int\frac{1}{k(t)}dt), \tag{17}$$

or in the original variables

$$a_{00}(t)=\frac{A_1 g(t)}{f(t)},\ f(t)=(h_1+h_0\int g(t)dt)g(t), \tag{18}$$

where h_0, h_1, and A_2 are constants.

As (15) can not be solved exactly, special cases will be considered

(i) When $C_0(x,t)=C_{00}(t)$, we find that $C_{00}(t)=-\frac{A_0^2}{h^2(t)}$

where A_0 is a constant.

Step 4. solve the auxiliary equation in (4), where $(4)_2$ solves to

$$\varphi(x,t)=\frac{4A_0^2w(t)-C_1(x,t)(h^2(t)e^{\frac{2A_0x}{h(t)}}+2A_0h(t)w(t))-2A_0h(t)e^{\frac{2A_0x}{h(t)}}}{2c_2(x,t)(h^2(t)e^{\frac{2A_0x}{h(t)}}+2A_0h(t)w(t))}, \tag{19}$$

where $w(t)$ is an arbitrary function, which is determined using $(4)_1$.

Step 5. Evaluate (3) to get a solution of (10) by

$$u(x,t)=\frac{(-4w_0^2A_0^2q^2(t)-e^{\frac{4A_0x}{h(t)}}h^2(t))(6A_0^2-A_1-h_0x)}{h(t)(2A_0w_0q(t)+h(t)e^{\frac{2A_0x}{h(t)}})^2}, +\frac{4A_0w_0q(t)h(t)e^{\frac{2A_0x}{h(t)}}(18A_0^2+A_1+h_0x)}{h(t)(2A_0w_0q(t)+h(t)e^{\frac{2A_0x}{h(t)}})^2}, \tag{20}$$

$$q(t)=e^{\int\frac{(-4A_0^3+2A_0A_1+h_0(h_1+h_0\int g(t)dt))g(t)}{(h_1+h_0\int g(t)dt)^2}dt},$$

where w_0 is a constant.

It is worth noticing that one can verify that the solution (given by (20)) satisfies (10).

(ii) By taking $C_0(x,t)=C_{00}(t)+\frac{C_{01}(t)}{x^2}$, and by a similar way, we find a second solution of (10);

$$u(x,t)=\frac{c_{00}x^2(-24c_{00}+w_0x-w_0x\cos(2(A+v(x,t))))g^2(t)}{2f(t)(\sqrt{c_{00}}x\,g(t)sin(A+v(x,t))+f(t)cos(A+v(x,t)))^2} +\frac{2(12c_{00}+w_0x)\cos^2(A+v(x,t))f^2(t)}{2f(t)(\sqrt{c_{00}}x\,g(t)sin(A+v(x,t))+f(t)cos(A+v(x,t)))^2} +\frac{2\sqrt{c_{00}}w_0x^2f(t)g(t)\sin(2(A+v(x,t)))}{2f(t)(\sqrt{c_{00}}x\,g(t)sin(A+v(x,t))+f(t)cos(A+v(x,t)))^2}, \tag{21}$$

$$v(x,t)=\frac{\sqrt{c_{00}}(-4c_{00}+w_0x)g(t)}{w_0f(t)},$$

where c_{00} is a constant.

Again, the solution (21) verifies the equation (10).

It worth noticing that during the computations, we have found that $c_2(x,t)$ and $c_1(x,t)$ are arbitrary functions. So that, we think that no loss of generality if we take $c_2(x,t)\equiv 1$, and $c_1(x,t)\equiv 0$.

We mention that if $c_{00}>0$ the solution in (21) gives rise to wave train solutions. While when $c_{00}<0$, it gives rise to soliton solutions.

The solution (21) is displayed in figures 1 (a) and (b), when $g(t)=1+t^2$.

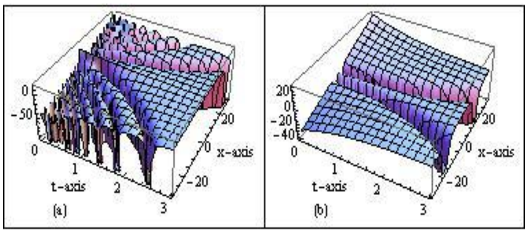

Fig 1: (a) a wave train solution when $c_{00}=1$,

(b) a ``winged''- soliton solution when $c_{00}=-1$

- Second case: When $k=3, n=4$, we follow the same steps as in the previous case.

In this case, two solutions were found which are given by;

$$u_1(x,t)=-\frac{12h_1^2w_0^2+(h_1-h_0x)(2h_1+w_0x)^2+(24h_1h_0w_0^2}{h(t)(2h_1+w_0x+(2h_0+w_0)\int g(t)dt)^2} +\frac{2(2h_0+w_0)(-h_1+h_0x)(2h_1+w_0x))(\int g(t)dt)}{h(t)(2h_1+w_0x+(2h_0+w_0)\int g(t)dt)^2} -\frac{(12h_0^2w_0^2-(2h_0+w_0)^2(-h_1+h_0x))(\int g(t)dt)^2}{h(t)(2h_1+w_0x+(2h_0+w_0)\int g(t)dt)^2}, \tag{22}$$

$$u_2(x,t)=\frac{4h_0x^7-144x^4h^2(t)-12h_0x^4h^3(t)(w_0+\frac{8}{h_0h(t)})}{h(t)(2x^3-3h^3(t)(w_0+\frac{8}{h_0h(t)}))^2} -\frac{432x\,h^5(t)(w_0+\frac{8}{h_0h(t)})-9x\,h_0h^6(t)(w_0+\frac{8}{h_0h(t)})^2}{h(t)(2x^3-3h^3(t)(w_0+\frac{8}{h_0h(t)}))^2} -\frac{432x\,h^5(t)(w_0+\frac{8}{h_0h(t)})-9x\,h_0h^6(t)(w_0+\frac{8}{h_0h(t)})^2}{h(t)(2x^3-3h^3(t)(w_0+\frac{8}{h_0h(t)}))^2}.$$

(23)

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The solutions (22) and (23) verify the equation (10).
The solutions in (22) and (23) that correspond to the case (i) and (ii) are displayed in figures 2 (a) and (b) respectively. In this figures $g(t)=1+t^2$.

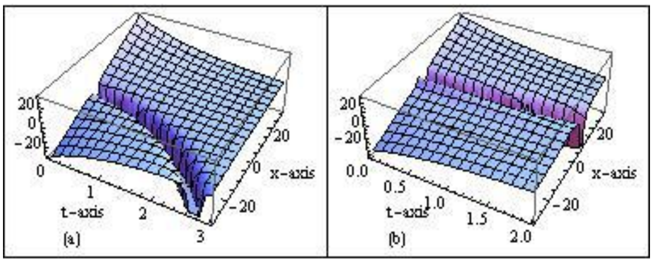

Fig 2: (a) case (i), (b) case(ii)

B. The rational function solutions

Here, we seek for rational function-solutionsof (10) where by using the condition (i) in lemma 2.2, we have two case; $r=n, k=1$ and $(m-1)(k-1)=n-r$. We confine ourselves to the case when $n=r$.
In this case the equation (4) becomes as

$$\phi_t = b_1(x,t)\phi + b_0(x,t), \phi_x = c_1(x,t)\phi + c_0(x,t), \quad (24)$$

together with the compatibility equation $\phi_{xt}=\phi_{tx}$.
We consider the Eq. (7), without loss of generality we take $d_n=1$, so that, we may write

$$u(x,t) = a_n(x,t) + \theta_{n-1}(\phi), \quad (25)$$

$$\theta_{n-1}(\phi) := \frac{P_{n-1}(\phi)}{Q_n(\phi)} = (\sum_{i=0}^{n-1} r_i(x,t)\phi^i)/(\phi^n + \sum_{i=0}^{n-1} d_i(x,t)\phi^i).$$

Lemma 3.1. a_n *satisfies KdV equation (10) if and only if* $\theta_{n-1,n}$ *verify the partial differential equation*

$$(\theta_{n-1})_t + f(t)(\theta_{n-1})_{xxx} + g(t)(\theta_{n-1})(\theta_{n-1})_x + (a_n\theta_{n-1})_x = 0. \quad (26)$$

It is worth noticing that for $n \neq 1$ we have a generalized $auto-B\ddot{a}cklund$ transformation, but for $n=1$ it is the $auto-B\ddot{a}cklund$ t transformation.

Theorem 3.2. *When* $n=1$, *a rational solution to (10) exists if and only if* $f(t)=cg(t)$, *where* c *is a constant.*

Proof. If $f(t)=cg(t)$ we use $\tau=\int g(t)dt$, $x=x$, and $u=u$, thus (10) becomes KdV equation with constant coefficients. It has a soliton solution in the variable $z=x-c_0\tau$, c_0 is a constant. This solution is rational in the exponential function that satisfies the auxiliary Eq. (24) when c_0 and c_1 are constants.
Now, we prove that if a rational solution exists then $f(t)=cg(t)$ by the converse statement. We assume that there exists $f(t)\neq cg(t)$ and a solution exists when $n=1$, as

$$u(x,t) = a_1(x,t) + \theta_0(\phi), \theta_0(\phi) = \frac{r_0(x,t)}{\phi + d_0(x,t)}, \quad (27)$$

where ϕ satisfies (24) , when $n=1$.

By taking $a_1(x,t)$ as a particular solution which is given in theorem 3.1, in this case, the principle equations, namely those arising from substituting into (26), we get $b_1(x,t), b_0(x,t)$ and two other equations; namely

$$A(x,t)(A_0(x,t) + d_0(x,t)B(x,t)) = 0,$$
$$A(x,t)(B_0(x,t) + B(x,t)) = 0, \quad (28)$$
$$A(x,t) = c_0(x,t) - d_0(x,t)c_1(x,t) + d_{0x}(x,t),$$

and $A_0(x,t)$, $B_0(x,t)$, and $B(x,t)$ are functions in $d_0, r_0, c_0, c_1, d_{0x}, d_{0xx}, \ldots,$ $A_0(x,t) - d_0(x,t)B(x,t) = A(x,t)$.
From the last result and $(24)_2$ when $n=1$, we find that;

$$\phi(x,t) + d_0(x,t) = H_0(t)e^{\int c_1(x,t)dx}. \quad (29)$$

Thus, the denominator is matched to the numerator in (27). The equation (28) has the unique solution $A(x,t)=0$. Consequently, no rational solution exists to (10) unless $f(t)=cg(t)$. □
The study of the case $n\geq 2$ is not straightforward and will not be considered here.

IV. CONCLUSION

In this paper, we suggested an extended unified method for finding exact solutions to evolution equations with variable coefficients. A wide class of exact solutions to KdV equation with time-dependent coefficients is obtained. Some of these solutions show ``winged'' soliton (anti-soliton) or wave train solutions. The method and the solutions that we obtained here are completely new and we can use this method to find exact solutions of coupled evolution equations. But in this case we think that parallel computations should be used.

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