Multi-objective Linear Programming Problems involving Fuzzy Parameters

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Abstract—In many Linear programming problems it becomes desirable to have multiple criterions being optimized under the similar stated constraints. Also in the real life model the data can rarely be determined exactly with certainty and precision. The experimental data or the experts' estimation may lead us to the interval of real numbers as the estimates of the parameters involved in the optimization problems. Such parameters can be efficiently modeled as fuzzy number. The situation can be then represented as Multi-objective LPP with fuzzy parameters. We here propose the method to compute the solution for multi-objective fully fuzzy LPP involving parameters represented by triangular fuzzy numbers.

Index Terms— fully fuzzy LPP, fuzzy numbers, multi - objective, triangular fuzzy numbers.

I. INTRODUCTION

Multi-objective optimization problem occurs in various real life applications such as Aerodynamic design, Medical decision making, Industrial neural network design etc. In many models the data can rarely be determined exactly with certainty and precision. We may consider the intervals of real numbers and be sure that the data fluctuates in these intervals. It is clear that precision in decision making is very important and any error may give rise to high expenses in application. That is why in such problems, the data is considered in the form of fuzzy numbers [1,2, 3, 4].

Bellman and Zadeh [5] proposed the concept of decision making in fuzzy environment. The first formulation of fuzzy linear programming was presented by Zimmerman [6]. A multi objective approach for fuzzy linear programming problems was proposed by Tanaka et al [7]. Pandian [8] used sum of objectives for solving multi-objective programming problems.

In this article, we propose a method to solve multi-objective linear problems with all the involved parameters, cost coefficients in the objective functions, technological coefficients in the constraints and the resources, being represented by fuzzy numbers. The problem is first converted into equivalent crisp form giving rise to multi-objective linear problem which is solved then using Pareto's optimality technique.

II. PRELIMINARIES

A. *Definition* To qualify as a fuzzy numbers, a fuzzy set A on R must possess at-least the following three properties:

(i) A must be a normal fuzzy set,
(ii) $^{\alpha}A$ must be a closed interval for every α ∈ (0,1],
(iii) The support of A, ^{0+}A, must be bounded.

B. *Definition* A fuzzy numbers $\tilde{a}$ is a triangular fuzzy number denoted by (s, l, r) where s, l, r are real numbers and its membership function $\mu_{\tilde{a}}(x) = A(x)$ is given as

$$A(x) = \begin{cases} \dfrac{x-s+l}{l} & \text{for } s-l < x \le s \\ \dfrac{s+r-x}{r} & \text{for } s < x \le s+r \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

We need the following definitions of the arithmetic operations on fuzzy numbers, for triangular fuzzy numbers $A = (s_1, l_1, r_1)$ and $B = (s_2, l_2, r_2)$.

C. *Definition* Summation and scalar multiplication are defined as:

$$(s_1, l_1, r_1) + (s_2, l_2, r_2) = (s_1 + s_2, l_1 + l_2, r_1 + r_2) \qquad (2)$$
$$a(s_1, l_1, r_1) = (as_1, al_1, ar_1), \qquad (3)$$

for any non-negative real number a.

D. *Definition* The partial order ≤ is defined by A ≤ B iff

$$\begin{aligned} s_1 &\le s_2 \\ s_1 - l_1 &\le s_2 - l_2 \\ s_1 + r_1 &\le s_2 + r_2 \end{aligned} \qquad (4)$$

E. *Theorem:* (Pareto optimality) For a problem with k objective functions, a design variable vector x^* is Pareto optimal if and only if there is no vector x in the feasible space with the characteristics, $f_i(x) \le f_i(x^*)$ for all 1,2,…,k and $f_i(x) \le f_i(x^*)$ for at least one i.

III. FUZZY LINEAR PROGRAMMING PROBLEMS

The fully fuzzy linear programming problems are given as:

$$\text{Maximize } \tilde{Z} = \sum_{j=1}^{n} \tilde{c}_j x_j$$

$$\text{s.t. } \sum_{j=1}^{n} \tilde{a}_{ij} x_j \le \tilde{b}_i, \quad i = 1, \ldots, m$$
$$x_j \ge 0, \qquad j = 1, \ldots, n \qquad (5)$$

where, $\tilde{c}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ are fuzzy numbers and x_j are fuzzy variables whose states are fuzzy numbers.

Considering the fuzzy parameters as triangular fuzzy numbers the problem (5) can be given as

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$$\text{Maximize } \tilde{Z}$$
$$\text{where, } \tilde{Z} = \sum_{j=1}^{n} (cs, cl, cr)_j x_j,$$
$$\text{s.t. } \sum_{j=1}^{n} (as, al, ar)_{ij} x_j \leq (bs, bl, br)_i, \quad i = 1, \dots, m$$
$$x_j \geq 0, \quad j = 1, \dots, n \tag{6}$$

where, $(cs, cl, cr)_j$ is the j^{th} fuzzy coefficient in the objective function, $(as, al, ar)_{ij}$ is the fuzzy coefficient of j^{th} variable in the i^{th} constraint, $(bs, bl, br)_i$ is the i^{th} fuzzy resource.

Thakre et al. solved such problem by converting (6) into to equivalent crisp multi-objective linear problem as given below

$$\text{Maximize } (Z_1, Z_2, Z_3)$$
$$\text{where, } \quad Z_1 = \sum_{j=1}^{n} cs_j x_j, \; Z_2 = \sum_{j=1}^{n} cl_j x_j, \; Z_3 = \sum_{j=1}^{n} cr_j x_j$$
$$\text{s.t. } \sum_{j=1}^{n} as_{ij} x_j \leq bs_i, \quad i = 1, \dots, m$$
$$\sum_{j=1}^{n} (as_{ij} - al_{ij}) x_j \leq (bs_i - bl_i), \quad i = 1, \dots, m$$
$$\sum_{j=1}^{n} (as_{ij} + ar_{ij}) x_j \leq (bs_i + br_i), \quad i = 1, \dots, m$$
$$x_j \geq 0, \quad j = 1, \dots, n$$

We here consider the multi-objective fully fuzzy LPP with the parameters as fuzzy numbers and propose the method similar to that of Thakre.

IV. MULTI-OBJECTIVE FUZZY LPP

The general form of Fuzzy Multi-Objective Linear Programming Problem is given as

$$\text{Maximize } \tilde{Z}^1, \tilde{Z}^2, \dots, \tilde{Z}^k$$
$$\text{where, } \tilde{Z}^k = \sum_{j=1}^{n} \tilde{c}_j^k x_j, \quad k = 1, \dots, K$$
$$\text{s.t. } \sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq \tilde{b}_i, \quad i = 1, \dots, m$$
$$x_j \geq 0, \quad j = 1, \dots, n \tag{7}$$

where, $\tilde{c}_{kj}$ $\tilde{a}_{ij}$ and $\tilde{b}_i$ are fuzzy numbers and x_j are fuzzy variables whose states are fuzzy numbers.

Considering the cost coefficients, the technological coefficients and the resources being represented by triangular fuzzy numbers (s, l, r), the Problem (7) can be rewritten as:

$$\text{Maximize } \tilde{Z}^1, \tilde{Z}^2, \dots, \tilde{Z}^k$$
$$\text{where, } \tilde{Z}^k = \sum_{j=1}^{n} (cs, cl, cr)_j^k x_j, \quad k = 1, \dots, K$$
$$\text{s.t. } \sum_{j=1}^{n} (as, al, ar)_{ij} x_j \leq (bs, bl, br)_i, \quad i = 1, \dots, m$$
$$x_j \geq 0, \quad j = 1, \dots, n \tag{8}$$

Where, $(cs, cl, cr)_j^k$ is the j^{th} fuzzy coefficient in the k^{th} objective function, $(as, al, ar)_{ij}$ is the fuzzy coefficient of j^{th} variable in the i^{th} constraint, $(bs, bl, br)_i$ is the i^{th} fuzzy resource.

V. SOLUTION TECHNIQUE FOR MOFLPP

For a multiple objective optimization problem with k fuzzy goals $\tilde{Z}^1, \tilde{Z}^2, \dots, \tilde{Z}^k$ and m fuzzy constraints, $\tilde{G}^i: \sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq bi, \quad i=1,\dots,m$. By generalizing the analogy from the single objective function, the resulting fuzzy decision is given as $\tilde{Z}^1 \cap \tilde{Z}^2 \cap \dots \cap \tilde{Z}^k \cap \tilde{G}^1 \cap \tilde{G}^2 \dots \cap \tilde{G}^m$. In terms of corresponding membership values for the fuzzy goals and the fuzzy constraints, the resulting decision is

$$\mu_{\tilde{D}}(X) = \min_{k,i} (\mu_{\tilde{Z}^k}(X), \mu_{\tilde{G}^i}(X))$$

An optimum solution X^* is one at which the membership function of the resulting decision $\tilde{D}$ is maximum. That is,

$$\mu_{\tilde{D}}(X^*) = \max \; \mu_{\tilde{D}}(X)$$

In our considered problem (8) the cost coefficients in the objective functions are represented by triangular fuzzy numbers. Using the method proposed by Thakre at el. for solving the fully fuzzy LPP with single objective function, each $\tilde{Z}^k$ in problem (8), give rise to three crisp objective functions, hence the converted problem will involve now $3k$ objective functions to be optimized as below

$$\text{Maximize } (Z_1^1, Z_2^1, Z_3^1, Z_1^2, \dots, Z_3^k)$$
$$\text{where, } \quad Z_1^k = \sum_{j=1}^{n} cs_j^k x_j; \; Z_2^k = \sum_{j=1}^{n} cl_j^k x_j; Z_3^k = \sum_{j=1}^{n} cr_j^k x_j$$
$$\text{for } k = 1, \dots, K$$
$$\text{s.t. } \sum_{j=1}^{n} as_{ij} x_j \leq bs_i, \quad i = 1, \dots, m$$
$$\sum_{j=1}^{n} (as_{ij} - al_{ij}) x_j \leq (bs_i - bl_i), \quad i = 1, \dots, m$$
$$\sum_{j=1}^{n} (as_{ij} + ar_{ij}) x_j \leq (bs_i + br_i), \quad i = 1, \dots, m$$
$$x_j \geq 0, \quad j = 1, \dots, n$$

Using Pareto method we form weighted objective function and hence the rewrite the problem as:

$$\text{Max } w_{11}Z_1^1 + w_{12}Z_2^1 + w_{13}Z_3^1 + w_{21}Z_1^2 + \cdots + w_{k3}Z_3^k$$

$$\text{where, } \quad Z_1^k = \sum_{j=1}^{n} cs_j^k x_j \; ; \; Z_2^k = \sum_{j=1}^{n} cl_j^k x_j \; ;$$

$$Z_3^k = \sum_{j=1}^{n} cr_j^k x_j \qquad k = 1,\dots,K$$

$$\text{s.t.} \quad \sum_{j=1}^{n} as_{ij} x_j \le bs_i, \quad i = 1,\dots,m$$

$$\sum_{j=1}^{n} (as_{ij} - al_{ij}) x_j \le (bs_i - bl_i), \quad i = 1,\dots,m$$

$$\sum_{j=1}^{n} (as_{ij} + ar_{ij}) x_j \le (bs_i + br_i), \quad i = 1,\dots,m$$

$$x_j \ge 0, \qquad j = 1,\dots,n \tag{9}$$

Solving the problem for different weights allows us to determine the solution.

VI. NUMERICAL EXAMPLE

Consider the Multi objective fuzzy Linear Problem

$$Maximize \;\; \tilde{Z}^1, \tilde{Z}^2$$

$$\text{where, } \; \tilde{Z}^1 = \tilde{c}_1 x_1 + \tilde{c}_2 x_2$$

$$\tilde{Z}^2 = \tilde{c}_3 x_1 + \tilde{c}_4 x_2$$

subject to the constraints

$$(3,2,1)x_1 + (6,4,1)x_2 \le (13,5,2)$$

$$(4,1,2)x_1 + (6,5,4)x_2 \le (7,4,2) \tag{10}$$

where the membership function of $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ and $\tilde{c}_4$ are

$$\mu_{\tilde{c}_1}(x) = \begin{cases} \frac{x-7}{3} & for\ 7 < x \le 10 \\ \frac{14-x}{4} & for\ 10 < x \le 14 \\ 0 & o.w. \end{cases},$$

$$\mu_{\tilde{c}_2}(x) = \begin{cases} \frac{x-20}{5} & for\ 20 < x \le 25 \\ \frac{35-x}{10} & for\ 25 < x \le 35 \\ 0 & o.w. \end{cases},$$

$$\mu_{\tilde{c}_3}(x) = \begin{cases} \frac{x-10}{4} & for\ 10 < x \le 14 \\ \frac{25-x}{9} & for\ 14 < x \le 25 \\ 0 & o.w. \end{cases},$$

$$\mu_{\tilde{c}_4}(x) = \begin{cases} \frac{x-25}{5} & for\ 25 < x \le 35 \\ \frac{40-x}{5} & for\ 35 < x \le 40 \\ 0 & o.w. \end{cases}$$

It is equivalent to solving the MOLPP

$$Maximize \begin{pmatrix} 7x_1 + 20x_2, 10x_1 + 25x_2, \\ 14x_1 + 35x_2, 25x_1 + 40x_2 \end{pmatrix}$$

subject to the constraints

$$\left.\begin{aligned} 3x_1 + 6x_2 &\le 13 \\ x_1 + 2x_2 &\le 8 \\ 4x_1 + 7x_2 &\le 15 \\ 4x_1 + 6x_2 &\le 7 \\ 3x_1 + x_2 &\le 3 \\ 6x_1 + 10x_2 &\le 9 \\ x_1, x_2 &\ge 0 \end{aligned}\right\} \tag{11}$$

That is, solving

$$Maximize \quad \begin{matrix} w_1(7x_1 + 20x_2) + w_2(10x_1 + 25x_2) \\ + w_3(14x_1 + 35x_2) + w_4(25x_1 + 40x_2) \end{matrix}$$

subject to the constraints (11) such that $\sum w_i = S$

Standard optimizing technique is used to solve the problem by using different weights in such a way that their sum is 2 and solving the resultant single objective LPP. For example by taking $w_1 = w_4 = 0$ and $w_2 = w_3 = 1$ we get MOLPP

$$Maximize \;\; (w) = 24x_1 + 60x_2$$

subject to the constraints (8)

Solving we get $(x_1^*, x_2^*) = (0, 0.9)$

Optimum solution is obtained as

$$f(x_1^*, x_2^*) = (0,0.9) = 0.9(\tilde{c}_2 + \tilde{c}_4)$$

$$\mu_{f(0,0.9)}(x) = \begin{cases} \frac{x - 40.5}{13.5} & for\ 40.5 < x \le 54 \\ \frac{67.5 - x}{13.5} & for\ 54 < x \le 67.5 \\ 0 & o.w. \end{cases}$$

Following table lists the solution for above MOLPP for various weights and it also shows that the solutions are independent of weights $(w_i, i = 1,2,3,4)$.

Sr. No	w_1	w_2	w_3	w_4	(x_1^*, x_2^*)
1	0	1	1	0	(0, 0.9)
2	0	1	0.5	0	(0, 0.9)
3	0.2	0.4	0.5	0.2	(0, 0.9)
4	0.1	0.2	0.3	0.4	(0, 0.9)
5	0	0.3	0	0.4	(0, 0.9)
6	0.2	0.4	0.6	0.8	(0, 0.9)
7	0.5	0	0.5	0	(0, 0.9)
8	0	1	1	0	(0, 0.9)
9	0	0	0	0.5	(0, 0.9)
10	0.3	0.1	1	1	(0, 0.9)
11	0.5	0.5	0.5	0.5	(0, 0.9)
12	0	0	0.5	0.5	(0, 0.9)
13	0.2	0.5	0.5	0.5	(0, 0.9)
14	0.1	0.2	0.3	0.4	(0, 0.9)
15	0	0.2	0	0.2	(0, 0.9)

VII. CONCLUSION

In this paper we have proposed the method for solving multi-objective fully fuzzy linear programming problem involving triangular fuzzy numbers and illustrated it with example.

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