Generation of Orthogonal Sequences by Walsh Sequences

Ahmad Hamza Al Cheikha, Ruchin Jain

Abstract— ***Walsh sequences of the order 2k, k positive integer, form an additive group generated by Rademacher sequences set of k-order. Except the zero sequences, Walsh formed an orthogonal set of*** $2^{l+k-2}, 2^{2k-2}, 2^{2^k+k-3}$.

The present work allows generating orthogonal sequences with length 2k , 22k, l.2k respectively, by using Walsh sequences and theirs complements.

Index Terms— ***Walsh sequences, Rademacher sequences, Orthogonal sequences.***

I. INTRODUCTION

In 1923, J.L. Walsh defined a system of orthogonal functions that is complete over the normalized interval (0,1). The method of specifying the Walsh functions of arbitrary order $N = 2^k$, k = 1,2,… had been a problem of considerable difficulty until the year 1970, when Byrnes and Swick showed that the Walsh functions could be obtained from Rademacher functions and from the solutions of certain differential equations. Byrnes and Swick considered the inherent symmetric properties of the Walsh functions of order N are as a set of functions denoted by $\{W_J(t), t \in (0,T), J = 0,1,\ldots,N-1\}$ Walsh sequences of order 2^k, which are generated by the binary representation of Walsh functions of order $N = 2^k$, form a group under 2 addition (addition group). The set of these sequences except W_0 forms orthogonal set. Tables 1, 2, and 3 show the sequences of order 2^2, 2^3 and 2^4 respectively [1,2,3].

The Walsh functions can be generated by any of the following methods:

1. Using Rademacher functions.
2. Using Hadamard matrices.
3. Exploiting the symmetry properties of Walsh functions.
4. Using division ring under 2^k addition [4,5,6,7].

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II. THE IMPORTANCE OF THIS RESEARCH AND ITS OBJECTIVES

Walsh functions (or sequences) are used widely as orthogonal sets in the forward and the inverse link of communications channels in the CDMA systems especially in the second (IS-95-CDMA), the third….(CDMA200,…), the pilot channels, the Sync channels, and the Traffics channels [1,6]. sets of orthogonal sequences with length .

$$2^{2k-2} \text{ and } 2^{2^k+k-3}$$

This work aims to generate 2^k and 2^{2k} respectively, by Walsh sequences and theirs complements.

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III. RESEARCH METHODS AND MATERIALS

The Rademacher sequences of order k = 3 are

R_0 = (0 0 0 0 0 0 0 0)
R_1 = (0 0 0 0 1 1 1 1)
R_2 = (0 0 1 1 0 0 1 1)
R_3 = (0 1 0 1 0 1 0 1)

The Rademacher sequences of order k = 4 are

R_1 = (0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1)
R_2 = (0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1)
R_3 = (0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1)
R_4 = (0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1)

Graphs of the Walsh functions of order 8 are given in Fig. 1

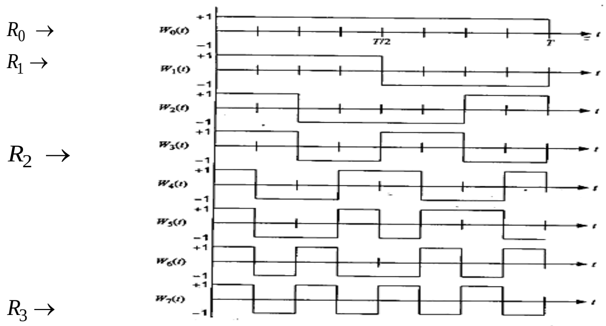

Figure 1. Walsh functions of order 8= 2^3

Tables 1, 2, 3 show set of Walsh sequences of order 2^2, 2^3 and 2^4 respectively.

Index Sequences	Walsh Sequences of order 4=2^2
00	W_0 = 0 0 0 0
01	W_1 = 0 0 1 1
10	W_2 = 0 1 1 0
11	W_3 = 0 1 0 1

Table 1. Walsh sequences of order 4 = 2^2

Index Sequences	Walsh Sequences of order 8=2^3
000	W_0 = 0 0 0 0 0 0 0 0
001	W_1 = 0 0 0 0 1 1 1 1
010	W_2 = 0 0 1 1 1 1 0 0
011	W_3 = 0 0 1 1 0 0 1 1
100	W_4 = 0 1 1 0 0 1 1 0
101	W_5 = 0 1 1 0 1 0 0 1
110	W_6 = 0 1 0 1 1 0 1 0
111	W_7 = 0 1 0 1 0 1 0 1

Table 2. Walsh sequences of order 8=2^3

Walsh Sequences of order 2^4=16
W_0 = (0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0)
W_1 = (0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1)
W_2 = (0 0 0 0 1 1 1 1 1 1 1 1 0 0 0 0)
W_3 = (0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1)
W_4 = (0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0)
W_5 = (0 0 1 1 1 1 0 0 1 1 0 0 0 0 1 1)
W_6 = (0 0 1 1 0 0 1 1 1 1 0 0 1 1 0 0)
W_7 = (0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1)
W_8 = (0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0)
W_9 = (0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1)
W_{10} = (0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0)
W_{11} =(0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1)
W_{12} =(0 1 0 1 1 0 1 0 0 1 0 1 1 0 1 0)
W_{13} =(0 1 0 1 1 0 1 0 1 0 1 0 0 1 0 1)
W_{14} =(0 1 0 1 0 1 0 1 1 0 1 0 1 0 1 0)
W_{15} =(0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1)

Table3. Walsh sequences of order 16=2^4

Definition 1

Suppose

$$y = (y_0, y_1, \ldots, y_{n-1}) \text{ and}$$
$$x = (x_0, x_1, \ldots, x_{n-1})$$

are vectors of length n on $GF(2)=\{0,1\}$. The autocorrelations function of x and y , denoted by $R_{x,y}$, is

$$R_{x,y} = \sum_{i=0}^{n-1} (-1)^{x_i + y_i}$$

where $x_i + y_i$ is computed mod 2. It is equal to the number of agreements components minus the number of disagreements corresponding to components [3]

Definition 2

Suppose G is a set of binary vectors of length n

$$G = \{X; X = (x_0, x_1, \ldots, x_{n-1}), x_i \in F_2 = \{0,1\}, i = \{0, \ldots, n-1\}\}$$

Let 1^*= -1 and 0^*=1. The set G is said to be orthogonal if the following two conditions are satisfied

$$\forall X \in G, \sum_{i=0}^{n-1} x_i^* \in \{-1,0,1\}, \quad or \quad |R_{x,0}| \le 1.$$

$$\forall X, Y \in G(X \ne Y), \sum_{i=0}^{n-1} x_i^* y_i^* \in \{-1,0,1\}, \quad or \quad |R_{x,y}| \le 1.$$

That is, the absolute value of "the number of agreements minus the number of disagreements" is equal to or less than 1 [8].

IV. RESULTS AND DISCUSSION

A. First Step

* The following matrix *A*, forms Walsh sequences of the order 2^3, after the zero sequence elimination.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The number of the orthogonal sequences sets (in this case), is

$$\binom{2^3-1}{0} = 1$$

* Each element of that matrix A can be substituted with its complement, to obtain complement of A. The matrix forms an orthogonal set of sequence.

If some row in the matrix A (suppose the first row) is substituted, with its complement which forms an orthogonal set. The matrix is obtained $A_{\overline{W_i}}$ ($A_{\overline{W_1}}$),

The number of the orthogonal sequences sets that could be formed (in this case), is

$$\binom{2^3-1}{1} = 2^3 - 1$$

$$A_{\overline{W_1}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

* If a family of rows is substituted in A (say the row $H = \{W_i \; ; i \in I \subseteq \{1, \ldots, 2^3 - 1\}\}$), then $A_{\overline{H}}$ forms an orthogonal set, because $\overline{H}$ is orthogonal set, The rows of matrix $A \setminus \overline{H}$ ($A \setminus \overline{H}$ is the rows set *A* without the rows set of $\overline{H}$)is orthogonal set, and rows of $A \setminus \overline{H}$ are orthogonal with each rows from $\overline{H}$.

Suppose that $Card\ \overline{H} = t \; ; 1 < t < 2^3 - 1,$ (where $Card\ \overline{H}$ is the number of rows $\overline{H}$).

Then the number of orthogonal sets that could be formed (in this case) is $\binom{2^3-1}{t}$, and the number the isomorphic sequences sets of Walsh-sequences of order k , is 2^{k-2} [7],

also for $k=3$, that order becomes 2^{3-2} = 2, and the number of the orthogonal sequences sets that could be generated from Walsh-sequences of order 2^3 or its isomorphic one, are

$2.2^{3-1} = 2^{(2^3)}$, each has a length 2^3

B Second Step

* The matrix A can be formed of Walsh- sequences rows of order 2^k , (after eliminating the zero – sequence), the number of the orthogonal sequences becomes $\binom{2^k-1}{0} = 1$ sets (in this case),

* If some row in the matrix A is substituted with its complement(suppose the row W_i),

Then $A_{\overline{W_i}}$, form an orthogonal set, because the sequence I of length 2^k and all its digits are 1.

Then, we have, $\overline{W_i}$ = $I + W_i$ and for $i \ne j$

$$\overline{W_i} + W_j = I + W_i + W_j = \overline{W_i + W_j} = \overline{W_k}$$

$\overline{W_i}$ and W_j are orthogonal sets.

And the number of the orthogonal sets that could be formed (in this case) is

$$\binom{2^k-1}{1} = 2^k - 1$$

If each element of the matrix *A* is substituted with its complement, which has rows complements to the rows of *A*.

The rows of $\overline{A}$ are orthogonal because for any: $i \ne j$, We have,

$$\overline{W_i} + \overline{W_j} = I + W_i + I + W_j = W_i + W_j = W_k$$

$\overline{W_i}$ *and* $\overline{W_j}$ are orthogonal sets.

and the number of the orthogonal sequences sets (in this case), becomes: $\binom{2^k-1}{2^k-1} = 1$

* If a family of rows of *A* is substituted with the complement. We have orthogonal set, for example for

$H = \{W_i\ ;\ i \in I \subseteq \{1,\ldots,2^k - 1\}\}$ then $A_{\overline{H}}$ forms an orthogonal set, because $A \setminus \overline{H}$ (the row of A without rows of H is an orthogonal set) is orthogonal set, $\overline{H}$ is orthogonal set and each row from $A \setminus \overline{H}$ is orthogonal with each row in $\overline{H}$, if $Card\ \overline{H} = t\ ;\ 1 < t < 2^k - 1$, where t is the number of rows of $\overline{H}$ (or H). Then the number of orthogonal sets that could be formed (in this case) is $\binom{2^k - 1}{t}$, and the number of orthogonal sequences sets that could be formed (considering the set of rows $\overline{A}$), is:

$$\sum_{t=0}^{2^k - 1} \binom{2^k - 1}{t} = 2^k - 1$$

Also, the number the isomorphic sequences sets of Walsh-sequences of order k, is 2^{k-2} [7], and the number of the orthogonal sequences sets that could be generated from Walsh-sequences of order 2^k or its isomorphic one, is : $2^{k-2}.2^{2^k - 1} = 2^{2^k + k - 3}$, and each has a length 2^k.

C. Third Step

* If we represent the Walsh – sequences set of order 2^k (assuming the sequence $W_i \neq W_0$), On substituting in each sequence of Walsh – sequences $W_j\ ;\ j = 0,\ldots,2^k$ each " **1** " with W_i and each " **0** " with $\overline{W_i}$, then we can find the following cases :

1. $\begin{array}{ccc} \mathbf{0} & \rightarrow & \overline{W_i} \\ \underline{+\mathbf{0}} & \rightarrow & \underline{+\overline{W_i}} \\ \mathbf{0} & & \underbrace{00\ldots0}_{2^k} \end{array}$

2. $\begin{array}{ccc} \mathbf{1} & \rightarrow & W_i \\ \underline{+\mathbf{1}} & \rightarrow & \underline{+W_i} \\ \mathbf{0} & & \underbrace{00\ldots0}_{2^k} \end{array}$

3. $\begin{array}{ccc} \mathbf{0} & \rightarrow & \overline{W_i} \\ \underline{+\mathbf{1}} & \rightarrow & \underline{+W_i} \\ \mathbf{1} & & \underbrace{11\ldots1}_{2^k} \end{array}$

4. $\begin{array}{ccc} \mathbf{1} & \rightarrow & W_i \\ \underline{+\mathbf{0}} & \rightarrow & \underline{+\overline{W_i}} \\ \mathbf{1} & & \underbrace{11\ldots1}_{2^k} \end{array}$

The resultant sequences are denoted by

$\mathbf{W}(W_i) = \{W_j(W_i)\ ;\ j = 0,\ldots,2^k\}$

Then the Walsh – sequences set, forms an associative group and each, the numbers of ones are equal to the number of zero, the set $\mathbf{W}$ (W_i) forms an orthogonal set. Also the $(\mathbf{W} \setminus W_0)(W_0)$ forms an orthogonal set. Similarly, the number of orthogonal sequences sets (in this case) is 2^k. Also the number of the isomorphic sequences sets of Walsh-sequences of order k, is 2^{k-2} [7], and the number of the orthogonal sequences sets that could be generated (in this case) is : $2^k.2^{k-2} = 2^{2k-2}$, each has a length 2^{2k}

D.Fourth Step

Let **W** be the Walsh – sequences set of order 2^k , and the binary sequence ω of length l . Substitute ω in each sequence of Walsh– sequences each " **1** " with ω and each " **0** " with $\overline{\omega}$, the resultant sequences be denoted as :

$$(\mathbf{W} \setminus W_0)(\omega) = \{W_j(\omega)\ ;\ j = 1,\ldots,2^k\}$$

Since the Walsh – sequences set, forms an associative group and each, the number of ones are equal to the number of zeros. Therefore, the sequences set $(\mathbf{W} \setminus W_0)(\omega)$, forms an orthogonal set of length $l.2^k$, and the number of orthogonal Sequences sets, that could be generated (depends on ω Variation) is

$$\sum_{t=0}^{t=l} \binom{l}{t} = 2^l$$

Also, the number of orthogonal sequences sets that can be generated (in this case) in the Walsh- sequences set or its isomorphic, is 2^{l+k-2}, each has a length $l.2^k$

Example

for k = 2, i = 1, we have :

$$\mathbf{W} = \begin{cases} W_0 = \mathbf{0} & \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \\ W_1 = \mathbf{0} & \mathbf{0} \quad \mathbf{1} \quad \mathbf{1} \\ W_2 = \mathbf{0} & \mathbf{1} \quad \mathbf{1} \quad \mathbf{0} \\ W_3 = \mathbf{0} & \mathbf{1} \quad \mathbf{0} \quad \mathbf{1} \end{cases}$$

$$\mathbf{W}(W_1) = \begin{cases} W_0(W_1) = 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0 \\ W_1(W_1) = 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1 \\ W_2(W_1) = 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0 \\ W_3(W_1) = 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1 \end{cases}$$

and for $\omega = 1101$, we have

$$(\mathbf{W} \setminus W_0)(\omega) = \begin{cases} W_1(\omega) = 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1 \\ W_2(\omega) = 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0 \\ W_3(\omega) = 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1 \end{cases}$$

V. RESULTS AND RECOMMENDATIONS

We generate $2^{2^k + k - 3}$ sets of orthogonal sequences with length 2^k. Also, 2^{2k-2} sets of orthogonal sequences with length 2^{2k} and 2^{l+k-2} sets of orthogonal sequences with length $l.2^k$. It is recommended to conduct a study for orthogonal sequences for implementing these new sequences. These sequences which can be used in CDMA systems that employ the Walsh sequence family for channel separation.

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REFERENCES

1. BEAUCHAMP, K. Applications of Walsh And Related Functions. Academic
2. WALSH, J.L. A Closed Set of Normal Orthogonal Functions. Amer. J. Math.
3. BYRNES, J.S.; SWICK. Instant Walsh Functions. SIAM Rever., VOL. 12 1970, 131.
4. LEE, D.S.; MILLER, L.E. CDMA Engineering Hand Book. Artech House, Boston,London, 1998. 48 - 70
5. MAC WILLIAMS, F.J.; SLOANE N.G.A. The Theory of Error-Correcting Codes. North-Holland, Amsterdam, 1978
6. YANG, S.C. CDMA RF System Engineering. Artech House, Boston and London, 1998. 80 - 110
7. AL-CHEIKHA, A. Generation of Sets of Sequences Isomorphic to Walsh Sequences. Qatar Uni. Sci. J. VOL 25, 2005. 16-30
8. YANG K, S. C.; KIM, Y.; KUMAR, P.V. Quasi-orthogonal Sequences for Codes-Division Multiple Access Systems. IEEE Trans. On Information Theory, VOL 46, 2000. 982-993

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