Some Aspects of the Geometry of Surfaces with a System of Flat Coordinate Lines

V.N. Ivanov, GIL-OULBE Mathieu

Abstract- The geometry of surfaces with system of flat coordinate lines is investigated. The general vector equation of surfaces obtained. On the basis of which formulas of the main quadratic forms are obtained. The condition at which performance the system of flat coordinate lines will be system of the main lines of curvature is obtained. The subclass of normal surfaces – surfaces with system of flat coordinate lines in the normal planes of directrix is studied. As an example on the basis of the general formulas the equation of Monge's ruled surfaces is obtained. Drawings of surfaces of Monge with various flat and spatial directrices and various flat generatrices are given.

Keywords: Surfaces with system of flat coordinate lines, the vector equation, coefficients of quadratic forms, lines of main curvatures, normal surfaces, Monge's ruled surfaces

I. INTRODUCTION

Calculation and designing of shells requires preliminary studying of geometry of surfaces on which shells are outlined. The general questions of geometry of surfaces are considered in differential geometry [1-4]. The majority of monographs and the textbooks devoted to methods of calculation of thin-walled structures are preceded by a summary of the general theory of surfaces. From private types of surfaces in existing literature surfaces of revolution, ruled surfaces and some other private types of surfaces most in detail are studied. However, when developing methods of calculation of specific classes of shells there is a need for more detailed studying of geometry of special class of surfaces. Variety of geometric forms is almost infinitely. In the encyclopedia of analytical surfaces [5], the equations and plotting of more than 500 different types of surfaces are given.

Researchers face a dilemma: to study separate, private types of surfaces, or to prefer research of wider classes of surfaces. As show practical researches it is necessary to combine both directions. investigations of a class of surfaces allows to obtain the most general formulas and properties which then can be used for studying narrower classes and groups of surfaces. At the same time studying of specific surfaces, allows to reveal some properties which then can be generalized and widespread on wider class of structures.

One of wide classes of surfaces is the class of surfaces with set of flat coordinate lines. This class treats the majority of surfaces used in practice: surfaces of revolution, transfer surface, ruled surfaces. A priori, by definition, the surfaces generated by movement of the flat line belong to this class of surfaces: cyclic surfaces, Monge's ruled surfaces, surfaces Joachimsthal, Dupin's cyclides.

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Dr. V.N. Ivanov, Dsc. civil engineering, professor
Dr. GIL-OULBE Mathieu, PhD civil engineering, associate professor

Practically any surface can be presented as consecutive positions of set of flat curves. The surface of revolution can be presented as movement of a circle of the variable radius moving along a straight line of centers, or as consecutive positions of the flat generatrix rotating around an axis. These representations are natural. However, to try to describe the same surface of rotation on the basis of the flat curves obtained by progress of the plane of the parallel line of centers, it is inexpedient. The analytical equations of coordinate lines received thus will be extremely difficult, and a surface on the basis of these lines will be difficult to reproduce.

The surface with family of flat coordinate lines can be divided into two subclasses:

1. The surfaces generated by movement of some flat curve (generatrix), according to any chosen law in space. Thus, at the movement the generating curve also is transformed under some law.
2. The surfaces generated by a flat curve, which ends move on two directrices curves. Between directing curves some law of compliance of coordinates of directing through which passes a generatrix is established. Generally generating curve can be transformed under some law.

The developable surfaces generated by movement of a straight line on two directrices can be an example of surfaces of the second subclass. Condition of compliance of coordinate points of directrices through which passes a directrix straight line, the condition is that tangents to directrices in reference points of a directrix lie in the general plane being the tangent plane of a torsion surface [6]. Koons's [7] surfaces, velaroidal surfaces belong to a subclass of surfaces of the second type [8], etc. In this work are considered a subclass of surfaces with system of flat coordinate lines of the first type.

The surface obtained by three-dimensional motion by the flat line which was transformed in the course of movement (fig.1), can be described as follows:

Let some spatial or flat directing curve be defined by $\boldsymbol{r}(u)$. The curve is accompanied by a Frenet trihedral ($\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$ - vectors of a tangent, a normal and a binormal). Let's set in each point of a directing curve a secant plane which position is defined by a vector of a single normal $\overline{n}(u)$ (fig. 1).

The vector of a normal can be presented, for example, in the form of decomposition on a Frenet trihedral by a directrix

$$\boldsymbol{n}(u)=(\boldsymbol{n\tau})\boldsymbol{\tau}+(\boldsymbol{n\nu})\boldsymbol{\nu}+(\boldsymbol{n\beta})\boldsymbol{\beta}\,. \qquad (1)$$

For the description of the equation of a generatrix in a secant plane are set basis vectors of rectangular system of coordinates $\boldsymbol{e}_0(u)$ and $\boldsymbol{g}_0(u)$ with the center on a directing curve. Together with a vector of unit normal to a secant plane $\boldsymbol{n}(u)$, these vectors make a mobile orthogonal simplex reference on a directing curve. Obviously, only two vector functions are independent $\boldsymbol{n}(u)$ and $\boldsymbol{e}_0(u)$. Thus, choosing the vector direction $\boldsymbol{e}_0(u)$ in a secant plane, it is possible to obtain, the most convenient superficial system of coordinates, for example, demanding orthogonality of obtained superficial coordinate system. The third vector function is defined from an orthogonality condition, as cross product $\boldsymbol{g}_0(u)=\boldsymbol{n}(u)\times\boldsymbol{e}_0(u)$.

In a secant plane the equation of the flat generatrix $R(u,v)$, which is transforming on certain law at movement of the plane along a directrix is set. The equation of a generatrix can be written, in particular, using the polar system of coordinates for what in a secant plane the vector equation of a circle of single radius is entered

$$\boldsymbol{e}(u,v)=\boldsymbol{e}_0(u)\cos v+\boldsymbol{g}_0\sin v, \qquad (2)$$

v is the polar angle counted in a secant plane from unitary

At differentiation on argument v, vector-functions mutually change to each other:

$$\frac{\partial \boldsymbol{e}}{\partial v}=-\boldsymbol{e}_0(u)\sin v+\boldsymbol{g}_0(u)\cos v=\boldsymbol{g}\,;\ \frac{\partial \boldsymbol{g}}{\partial v}=-\boldsymbol{e}\,; \qquad (4)$$

Taking into account the designations the surface equation with system of flat coordinate lines (fig.1) in a vector form can be written down as

$$\boldsymbol{\rho}(u,v)=\boldsymbol{r}(u)+R(u,v)\boldsymbol{e}(u,v), \qquad (5)$$

$\boldsymbol{\rho}(u,v)$ is the surface's radius-vector;

Let's note the properties of vector functions $\boldsymbol{e}(u,v)$, $\boldsymbol{g}(u,v)$, $\boldsymbol{n}(u)$. As these vector functions are mutually orthogonal vectors of single length, then:

$$(\boldsymbol{ee})=(\boldsymbol{g\,g})=(\boldsymbol{nn})=1\,;$$

$$(\boldsymbol{eg})=(\boldsymbol{en})=(\boldsymbol{gn})=0\,;$$

$$(\boldsymbol{e}\times\boldsymbol{g})=\boldsymbol{e}\,;\ (\boldsymbol{n}\times\boldsymbol{e})=\boldsymbol{g}\,;\ \boldsymbol{g}\times\boldsymbol{g}=\boldsymbol{e}\,. \qquad \textbf{(6)}$$

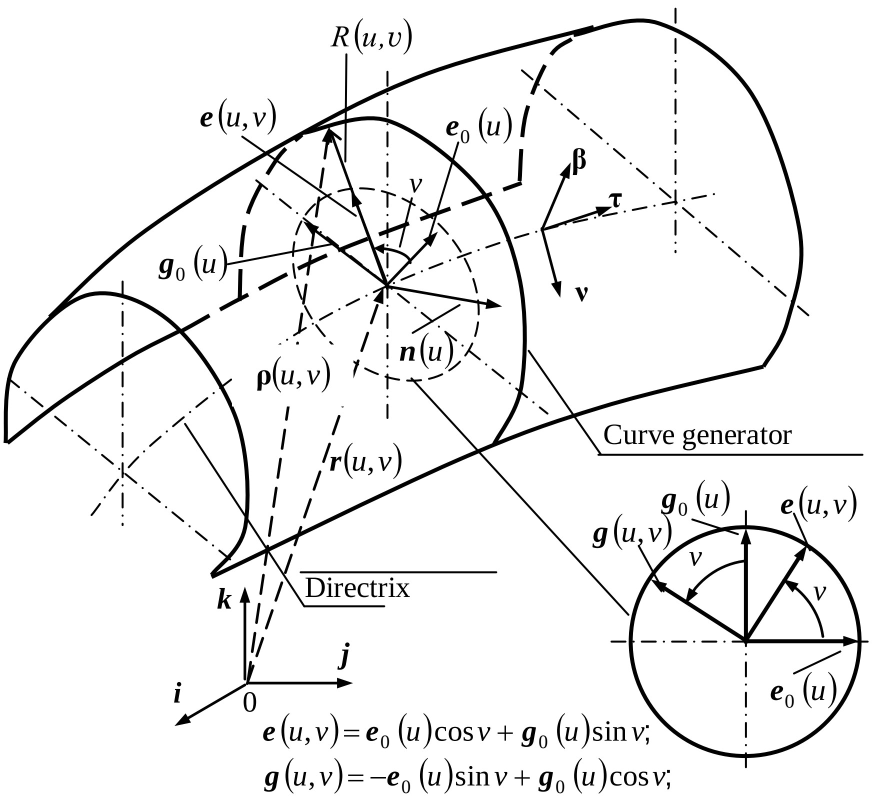

Fig.1. Surface with system of flat coordinate lines

vector $\boldsymbol{e}_0(u)$ in the direction of unitary vector $\boldsymbol{g}_0(u)$.

The vector $\boldsymbol{g}(u,v)v=-\boldsymbol{e}_0(u)\sin v+\boldsymbol{g}_0\cos v$, orthogonal to vector $\boldsymbol{e}(u,v)$, lies in a secant of the plane too and also is the equation of a circle of single radius. Vectors $\boldsymbol{e}(u,v)$, $\boldsymbol{g}(u,v)$ rotating round a normal vector, make together with a normal vector a mobile reference point on a directrix $\boldsymbol{r}(u)$. As well as a normal vector, they can be presented in the form of decomposition on Frenet trihedral:

$$\boldsymbol{e}(u,v)=(\boldsymbol{e\tau})\boldsymbol{\tau}+(\boldsymbol{e\nu})\boldsymbol{\nu}+(\boldsymbol{e\beta})\boldsymbol{\beta}\,;$$

$$\boldsymbol{g}(u,v)=(\boldsymbol{g\tau})\boldsymbol{\tau}+(\boldsymbol{g\nu})\boldsymbol{\nu}+(\boldsymbol{g\beta})\boldsymbol{\beta}, \qquad (3)$$

as well as on the contrary, vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$ can be presented in the form of decomposition according to simplex reference $\boldsymbol{e}, \boldsymbol{g}, \boldsymbol{n}$.

$$(\boldsymbol{ee'})=(\boldsymbol{gg'})=(\boldsymbol{nn'})=0\,;(\boldsymbol{e\dot{e}})=(\boldsymbol{g\dot{g}})0\,;$$

$$(\boldsymbol{e'g})=-(\boldsymbol{eg'});$$

$$(\boldsymbol{\dot{e}g})=-(\boldsymbol{e\dot{g}});\ (\boldsymbol{e'n})=-(\boldsymbol{en'});\ (\boldsymbol{g'n})=-(\boldsymbol{gn'}); \qquad (7)$$

Hereinafter for convenience of notation we will use designations of partial derivatives on arguments *u*, *v* -

$$\frac{\partial f}{\partial u}=f_u=f' \quad \text{и} \quad \frac{\partial f}{\partial v}=f_v=\dot{f}\,.$$

By using formulas (4), (6), (7), we obtain

$$(\boldsymbol{e'g})=[\boldsymbol{e}_0'\cos v+\boldsymbol{g}_0\sin v]\cdot[-\boldsymbol{e}_0\sin v+\boldsymbol{g}_0\cos v]=$$

.

$$=(\boldsymbol{e}_0'\boldsymbol{g}_0)\cos^2 v-(\boldsymbol{e}_0\boldsymbol{g}_0')\sin^2 v=(\boldsymbol{e}_0'\boldsymbol{g}_0)$$

(8)

The properties described by formulas (4), (6) - (8), show expediency of introduction by a vector-function $\boldsymbol{e}(u,v)$, $\boldsymbol{g}(u,v)$.

For the first time the equation of a vector function of a single circle in any plane was used in investigation of the geometry of cyclic surfaces [9, 10]. Vector-functions of a circle of single radius were usually used in the horizontal plane for rotational surfaces. Introduction of vector-functions of circles of single radius in any plane allowed investigating the geometry of cyclic surfaces in a general aspect. Attempts of many authors to conduct similar researches in a scalar form lead to bulky formulas and don't allow analyzing the obtained results effectively. Synthesis of researches on the geometry of the cyclic allowed passing to studying of the geometry of surfaces with a set of flat coordinate lines of a general aspect [11-17].

Formulas (6), (7) allow obtaining a decomposition of derivatives of vector-functions, on argument u on a simplex reference $\boldsymbol{e}(u,v)$, $\boldsymbol{g}(u,v)$, $\boldsymbol{n}(u)$, using derivatives of only initial (set) functions $\boldsymbol{n}(u)$ and $\boldsymbol{e}_0(u)$:

$$\boldsymbol{e}' = (\boldsymbol{e}'\boldsymbol{g})\boldsymbol{g} + (\boldsymbol{e}'\boldsymbol{n})\boldsymbol{n} = (\boldsymbol{e}_0'\boldsymbol{g}_0)\boldsymbol{g} - (\boldsymbol{e}\boldsymbol{n}')\boldsymbol{n}\cdot;$$
$$\boldsymbol{g}' = (\boldsymbol{e}\boldsymbol{g}')\boldsymbol{e} + (\boldsymbol{g}'\boldsymbol{n})\boldsymbol{n} = -(\boldsymbol{e}_0'\boldsymbol{g}_0)\boldsymbol{e} - (\boldsymbol{g}\boldsymbol{n}')\boldsymbol{n};$$
$$\boldsymbol{n}' = (\boldsymbol{e}\boldsymbol{n}')\boldsymbol{e} + (\boldsymbol{g}\boldsymbol{n}')\boldsymbol{g};$$
$$\boldsymbol{e}'' = -\left[(\boldsymbol{e}_0\boldsymbol{g}_0)^2 + (\boldsymbol{e}\boldsymbol{n}')^2\right]\boldsymbol{e} - [(\boldsymbol{e}\boldsymbol{n}'') + 2(\backslash\,\boldsymbol{e}_0\boldsymbol{g}_0)\boldsymbol{g}(\bar{g}\boldsymbol{n}')]\boldsymbol{n} +$$
$$+ [(\boldsymbol{e}_0''\boldsymbol{g}_0) + (\boldsymbol{e}_0\boldsymbol{n})(\boldsymbol{g}_0\boldsymbol{n}') - (\boldsymbol{e}\boldsymbol{n}')(\boldsymbol{g}\boldsymbol{n}')]\boldsymbol{g};$$
$$\boldsymbol{g}'' = -[(\boldsymbol{e}_0''\boldsymbol{g}_0) + (\boldsymbol{e}_0\boldsymbol{n}')(\boldsymbol{g}_0\boldsymbol{n}') + (\boldsymbol{e}\boldsymbol{n}')(\boldsymbol{g}\boldsymbol{n}')]e -$$
$$-\left[(\boldsymbol{e}_0'\boldsymbol{g}_0)^2 + (\boldsymbol{e}\boldsymbol{n}')^2\right]\boldsymbol{g} - [(\boldsymbol{g}\boldsymbol{n}'') + 2(\boldsymbol{e}_0'\boldsymbol{g}_0)(\boldsymbol{e}\boldsymbol{n}')]\boldsymbol{n}; \quad (9)$$

By differentiating the vector-equation of a surface (2) on arguments u, v, decomposing a vector on a simplex reference and using formulas (9), we obtain

$$\boldsymbol{\rho}_u = T_{11}\boldsymbol{e} + T_{12}\boldsymbol{g} + T_{13}\boldsymbol{n}; \quad \boldsymbol{\rho}_v = T_{21}\boldsymbol{e} + T_{22}\boldsymbol{g} + T_{23}\boldsymbol{n};$$
$$\boldsymbol{\rho}_{uu} = T_{31}\boldsymbol{e} + T_{32}\boldsymbol{g} + T_{33}\boldsymbol{n}; \quad \boldsymbol{\rho}_{uv} = T_{41}\boldsymbol{e} + T_{42}\boldsymbol{g} + T_{43}\boldsymbol{n};$$
$$\boldsymbol{\rho}_{vv} = T_{51}\boldsymbol{e} + T_{52}\boldsymbol{g} + T_{53}\boldsymbol{n}, \quad (10)$$

где $T_{11} = s'(\boldsymbol{\tau e}) + R'$; $T_{12} = s'(\boldsymbol{\tau g}) + R(\boldsymbol{e}_0'\boldsymbol{g}_0)$;
$T_{13} = s'(\tau\boldsymbol{n}) - R(\boldsymbol{e}\boldsymbol{n}')$;

$T_{21} = \dot{R}$; $T_{22} = R$; $T_{23} = 0$;

$$T_{31} = s''(\boldsymbol{\tau e}) + s'k_s(\boldsymbol{e\nu}) - R\left[(\boldsymbol{e}_0'\boldsymbol{g}_0)^2 + (\boldsymbol{e}\boldsymbol{n}')^2\right] + R'';$$
$$T_{32} = s''(\tau\boldsymbol{g}) + s'k_s(\boldsymbol{g\nu}) + 2R'(\boldsymbol{e}_0'\boldsymbol{g}_0) + +R[(\boldsymbol{e}_0''\boldsymbol{g}_0) + (\boldsymbol{e}_0\boldsymbol{n}')(\boldsymbol{g}_0\boldsymbol{n}') - (\boldsymbol{e}\boldsymbol{n}')(\boldsymbol{g}\boldsymbol{n}')];$$
$$T_{33} = s''(\boldsymbol{n\tau}) + s'k_s(\boldsymbol{n\nu}) - 2R'(\boldsymbol{e}\boldsymbol{n}') - R[(\boldsymbol{e}\boldsymbol{n}'') + 2(\boldsymbol{e}_0'\boldsymbol{g}_0)(\boldsymbol{g}\boldsymbol{n}')];$$
$$T_{41} = \dot{R}' - R(\boldsymbol{e}_0'\boldsymbol{g}_0); \quad T_{42} = R' + \dot{R}\cdot(\boldsymbol{e}_0'\boldsymbol{g}_0);$$
$$T_{43} = -[R(\boldsymbol{g}\boldsymbol{n}') + \dot{R}(\boldsymbol{e}\boldsymbol{n}')]; \quad T_{51} = \ddot{R} - R; \quad T_{52} = 2\dot{R};$$
$$T_{53} = 0. \quad (11)$$

$s' = |\boldsymbol{r}'| = \left|\frac{d\boldsymbol{r}}{du}\right|$; $k_s = s'k$; k - the curvature of directrix $\boldsymbol{r}(u)$;.

On the basis of formulas of differential geometry taking into account formulas (10) we obtain coefficients of quadratic forms of surface.

A. Coefficients of the first quadratic form:

$$E = (\boldsymbol{\rho}_u\boldsymbol{\rho}_u) = T_{11}^2 + T_{12}^2 + T_{13}^2;$$
$$G = (\boldsymbol{\rho}_v\boldsymbol{\rho}_v) = T_{21}^2 + T_{22}^2 = R^2 + \dot{R}^2;$$
$$F = (\boldsymbol{\rho}_u\boldsymbol{\rho}_v) = T_{11}\cdot T_{21} + T_{12}\cdot T_{22}. \quad (12)$$

B. Discriminant of surface

$$\Sigma^2 = EG - F^2 = |\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v|^2 = G\cdot T_{13}^2 + d_{12}^2;$$
$$d_{12} = T_{11}\cdot T_{22} - T_{12}\cdot T_{21}; \quad (13)$$

C. Vector of a single normal to surface

$$\boldsymbol{m} = \frac{\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v}{|\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v|} = \frac{(-T_{22}\boldsymbol{e} + T_{21}\boldsymbol{g})\cdot T_{13} + d_{12}\boldsymbol{n}}{\Sigma}. \quad (14)$$

D. Coefficients of the second quadratic form:

$$L = (\boldsymbol{\rho}_{uu}\boldsymbol{m}) = \frac{(T_{21}T_{32} - T_{22}T_{31})\cdot T_{13} + d_{12}T_{33}}{\Sigma};$$
$$M = (\boldsymbol{\rho}_{uv}\boldsymbol{m}) = \frac{(T_{21}T_{42} - T_{22}T_{41})\cdot T_{13} + d_{12}T_{43}}{\Sigma} = \frac{[G(\bar{e}_0'g_0) - \phi_2]T_{13} + d_{12}T_{43}}{\Sigma};$$
$$N = (\boldsymbol{\rho}_{vv}\boldsymbol{m}) = \frac{(T_{21}T_{52} - T_{22}T_{51})T_{13}}{\Sigma} = \frac{(G - \phi_1)T_{13}}{\Sigma}, \quad (15)$$
$$\phi_1 = (R\ddot{R} - \dot{R}^2); \quad \phi_2 = R\dot{R}' - \dot{R}R'.$$

Formulas (12)-(15) submit geometrical characteristics of surfaces with family of flat coordinate lines of the most general type. It is extremely wide class of surfaces. From formulas (12), (15) it is visible that, generally, coefficients *F*, *M* aren't equal to zero, and, therefore, the obtained system of coordinates of a surface isn't neither orthogonal, nor conjugated, and families of the coordinate lines *u=const*, *v=const* aren't generally lines of main curvatures of surfaces.

Equating zero formula of coefficients *F* or *M* it is possible to obtain ratios for surface parameters at which the system of coordinates of a surface will be accordingly orthogonal or conjugated.

Important subclass of studied surfaces is a subclass of surfaces with system of the flat generatrix coordinate lines being lines of main curvature. The system of equations of the thin wall shells, written down for curvature lines, has a simplest form. When using numerical methods of calculation the orthogonal tangential system of coordinates also is the most convenient.

In order for generatrices - set of the coordinate *u=const* lines to be a lines of main curvature of surfaces, the ratio between coefficients of quadratic forms has to be carried out as:

$$MG - NF = 0. \quad (16)$$

By reducing in a ratio (16) general denominator (discriminant Σ) taking into account formulas of coefficients (12), (15), we obtain

$$\{[G(\boldsymbol{e}_0'\boldsymbol{g}_0) - \phi_2]G - (G - \phi_1)F\}T_{13} + Gd_{12}T_{43} = 0. \quad (17)$$

By substituting in the ratio (17) formulas of coefficients T_{ij} and introducing a number of designations, we have

$$F = T_{11}T_{21} + T_{12}T_{22} = \dot{R}[s'(\boldsymbol{\tau e}) + R'] + R[s'(\boldsymbol{\tau g}) + R(\boldsymbol{e}_0'\boldsymbol{g}_0)] = F_1 + F_2;$$
$$F_1 = [R^2(\boldsymbol{e}_0'\boldsymbol{g}_0) + \dot{R}R']; \quad F_2 = s'[\dot{R}(\boldsymbol{\tau e}) + R(\boldsymbol{\tau g})];$$
$$d_{12} = T_{11}T_{22} - T_{12}T_{21} = R[s'(\boldsymbol{\tau e}) + R'] - \dot{R}[s'(\boldsymbol{\tau g}) + R(\boldsymbol{e}_0'\boldsymbol{g}_0)] = d_1 + d_2;$$
$$d_1 = R[R' - \dot{R}(\boldsymbol{e}_0'\boldsymbol{g}_0)];$$
$$d_2 = s'[R(\boldsymbol{\tau e}) - \dot{R}(\boldsymbol{\tau g})];$$

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$$f=\left[G\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right)-\phi_2\right]G-\left(G-\phi_1\right)F_1=$$
$$=\left[G\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right)-\left(R\dot{R}'-\dot{R}R'\right)\right]G-\left(G-\phi_1\right)\left[R^2\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right)+\dot{R}R'\right]=$$
$$=\left[G\left(G-R^2\right)+R^2\left(R\ddot{R}-\dot{R}^2\right)\right]\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right)-GR\dot{R}'+\phi_1\dot{R}R'=$$
$$=\left(\dot{R}^4+R^3\ddot{R}\right)\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right)-GR\dot{R}'+\dot{R}R'\phi_1 .$$

Grouping items, the condition (17) will be written down as:

$$\Psi_1\cdot\left(\boldsymbol{e}\boldsymbol{n}'\right)+\Psi_2\cdot\left(\boldsymbol{g}\boldsymbol{n}'\right)\cdot+\Psi_3\cdot s'\cdot\left(\boldsymbol{\tau}\boldsymbol{n}\right)=0 , \qquad (18)$$

где $\Psi_1=-\left\{R\left[f-\left(G-\phi_1\right)F_2\right]+\dot{R}G\left(d_1+d_2\right)\right\};$

$\Psi_2=-RG\,d_{12};$

$\Psi_3=s'\left[f-\left(G-\phi_1\right)F_2\right].$

$$\psi_{1,1}=-R\,f-\dot{R}Gd_1=-R\left[\left(\dot{R}^4+R^3\ddot{R}\right)\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right)-GR\dot{R}'+\dot{R}R'\phi_1\right]-R\dot{R}G\left[R'-\dot{R}\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right)\right]=$$
$$=R\left\{\left[\dot{R}^2G-\left(\dot{R}^4+R^3\ddot{R}\right)\right]\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right)+\left[G\left(R\dot{R}'-\dot{R}R'\right)-\dot{R}R'\phi_1\right]\right\}=$$
$$=R\left\{G\phi_2-\left[\dot{R}R'+R^2\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right)\right]\phi_1\right\};$$
$$\psi_{1,2}=R\left(G-\phi_1\right)F_2-\dot{R}Gd_2=R\left[G-\phi_1\right]s'\left[\dot{R}\left(\boldsymbol{\tau}\boldsymbol{e}\right)+R\left(\boldsymbol{\tau}\boldsymbol{g}\right)\right]-\dot{R}Gs'\left[R\left(\boldsymbol{\tau}\boldsymbol{e}\right)-\dot{R}\left(\boldsymbol{\tau}\boldsymbol{g}\right)\right]=$$
$$-\left[Rs'\left(\boldsymbol{\tau}\boldsymbol{e}\right)\right]^{\bullet}R\phi_1+s'\left(\boldsymbol{\tau}\boldsymbol{g}\right)G^2;$$
$$\Psi_1=\psi_{1,1}+\psi_{1,2}=R\left\{G\phi_2-\left[\dot{R}R'+R^2\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right)\right]\phi_1\right\}-\left[Rs'\left(\boldsymbol{\tau}\boldsymbol{e}\right)\right]^{\bullet}R\phi_1+s'\left(\boldsymbol{\tau}\boldsymbol{g}\right)G^2=$$
$$=-R\left\{\dot{R}R'+R^2\left(\mathbf{e}_0'\mathbf{g}_0\right)+\left[Rs'\left(\boldsymbol{\tau}\mathbf{e}\right)\right]^{\bullet}\right\}\phi_1+G\left[R\phi_2+s'\left(\boldsymbol{\tau}\mathbf{g}\right)G\right]$$
;
$$\Psi_2=-RGd_{12}=-RG\left\{R\left[R'-\dot{R}\left(\mathbf{e}_0'\mathbf{g}_0\right)\right]-\left[Rs'\left(\boldsymbol{\tau}\mathbf{g}\right)\right]^{\bullet}\right\};$$
$$\Psi_3=\left[f-\left(G-\phi_1\right)F_2\right]=\left(\dot{R}^4+R^3\ddot{R}\right)\left(\mathbf{e}_0'\mathbf{g}_0\right)-GR\dot{R}'+\dot{R}R'\phi_1-\left[G-\phi_1\right]\left[Rs'\left(\boldsymbol{\tau}\boldsymbol{e}\right)\right]^{\bullet};$$
$$\phi_1=R\ddot{R}-\dot{R}^2=R^{2\cdot}\left(\frac{\dot{R}}{R}\right)^{\bullet};$$
$$\phi_2=R\dot{R}'-\dot{R}R'=R^{2\cdot}\left(\frac{\dot{R}}{R}\right)'=R^2\left(\frac{R'}{R}\right)^{\bullet};$$
$$\left[Rs'\left(\boldsymbol{\tau}\boldsymbol{e}\right)\right]^{\bullet}=s'\left(\boldsymbol{\tau}\boldsymbol{e}\right)\dot{R}+s'\left(\boldsymbol{\tau}\boldsymbol{g}\right)R;$$
$$\left[Rs'\left(\boldsymbol{\tau}\mathbf{e}\right)\right]^{\bullet}=-s'\left(\boldsymbol{\tau}\boldsymbol{e}\right)R+s'\left(\boldsymbol{\tau}\boldsymbol{g}\right)\dot{R}. \qquad (19)$$

The condition (18) represents quite difficult ratio which is difficult to be analyzed in a general view. But it can be used in investigation of various subclasses of surfaces.

Let's consider a subclass of normal surfaces with a set of flat coordinate lines.

Normal surfaces we will call surfaces, the system of flat generatrices which coordinate lines lies in the normal planes of the directrix. For example surfaces of revolution and Monge's ruled surfaces belong to this class of surfaces. For such surfaces the normal of secant plane coincides with the tangent directrix $\boldsymbol{n}=\boldsymbol{\tau}$, and, therefore,

$$\left(\boldsymbol{\tau}\boldsymbol{e}\right)=\left(\boldsymbol{\tau}\boldsymbol{g}\right)=0;\ \boldsymbol{n}'=\boldsymbol{\tau}'=s'k\,\mathbf{v}=k_s\mathbf{v};$$
$$\boldsymbol{n}''=\left(k_s\mathbf{v}\right)'=-s'k_s\boldsymbol{\tau}+k_s'\mathbf{v}+k_s\chi_s\boldsymbol{\beta};$$
$$\left(\boldsymbol{e}\boldsymbol{n}'\right)=k_s\left(\boldsymbol{e}\mathbf{v}\right);\ \left(\boldsymbol{g}\boldsymbol{n}'\right)=k_s\left(\boldsymbol{g}\mathbf{v}\right);$$
$$\left(\boldsymbol{e}\boldsymbol{n}''\right)=k_s'\left(\boldsymbol{e}\mathbf{v}\right)+k_s\chi_s\left(\boldsymbol{e}\boldsymbol{\beta}\right). \qquad (20)$$

Here k, χ are curvature and torsion of a directrix. $k_s=s'k$; $\chi_s=s'\chi$.

Introducing the parameters, k_s, χ_s is expedient as these combinations are usually simpler, than separately curvature and torsion, and, therefore, their introduction simplifies further transformations. The system of coefficients for normal surfaces will register as:

$$T_{11}=R';\ T_{12}=R\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right);\ T_{13}=s'-Rk_s\left(\boldsymbol{e}\mathbf{v}\right);\ T_{21}=\dot{R};$$
$$T_{22}=R;\ T_{23}=0;$$
$$T_{31}=s'k_s\left(\boldsymbol{e}\mathbf{v}\right)-R\left[\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right)^2+k_s^2\left(\boldsymbol{e}\mathbf{v}\right)^2\right]+R'';$$
$$T_{32}=s'k_s\left(\boldsymbol{g}\mathbf{v}\right)+2R'\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right)+R\left\{\left(\boldsymbol{e}_0''\boldsymbol{g}_0\right)+k_s^2\left[\left(\boldsymbol{e}_0\mathbf{v}\right)\left(\boldsymbol{g}_0\mathbf{v}\right)-\left(\boldsymbol{e}\mathbf{v}\right)\left(\boldsymbol{g}\mathbf{v}\right)\right]\right\};$$
$$T_{33}=s''-2k_sR'\left(\boldsymbol{e}\mathbf{v}\right)-R\left[k_s'\left(\boldsymbol{e}\mathbf{v}\right)+k_{s\ s}\left(\boldsymbol{e}\boldsymbol{\beta}\right)+2k_s\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right)\left(\boldsymbol{g}\mathbf{v}\right)\right];$$
$$T_{41}=\dot{R}'-R\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right); \qquad T_{42}=R'+\dot{R}\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right);$$
$$T_{43}=-k_s\left[R\left(\boldsymbol{g}\mathbf{v}\right)+\dot{R}\left(\boldsymbol{e}\mathbf{v}\right)\right];$$
$$T_{51}=\ddot{R}-R;\ T_{52}=2\dot{R};\ T_{53}=0. \qquad (21)$$

The condition (2.1.3) (flat generatrices are lines of curvature of a surface) for normal surfaces becomes:

$$k_sR\left\{\left[\dot{R}R'-R^2\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right)\right]\phi_1+G\phi_2\right\}\left(\boldsymbol{e}\mathbf{v}\right)-k_sR^2G\left[R'-\dot{R}\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right)\right]\left(\boldsymbol{g}\mathbf{v}\right)+$$
$$+s'\left[\left(\dot{R}^4+R^3\ddot{R}\right)\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right)-GR\dot{R}'+\dot{R}R'\phi_1\right]=0. \qquad (22)$$

The obtained condition significantly depends on types of directrices and generatrices and nature of change of a generatrix at movement along a directrix. Obviously, the condition (22) can significantly become simpler if nature of change of generatrices depends only on one parameter *u* or*v*. As an example of use of the above obtained results we will consider Monge's ruled surfaces.

G. Monge defined the surfaces generated by movement of a flat curve, some being developed surface lying in the tangent plane which rolls (or drives) without sliding on a being developed surface [17] into consideration. Surfaces are called ruled surface and in literature the name Monge's ruled surfaces is often used. Any point of the tangent plane of the being developed surface, makes when swing plane movement orthogonal to the provision of the tangent plane at every moment of movement. Therefore, the flat generatrix lies in the normal plane of the curve described by any point, lying in the same tangent plane as a generatrix. Therefore, Monge's ruled surface is a normal surface.

As the generating curve of a ruled surface doesn't change in the course of movement along a directrix $R=R(\upsilon)$, $R'=R''=\dot{R}'=0$, $\phi_2=0$.

Let's consider a condition at which performance generatrices of surfaces will be lines of main curvatures of surfaces. By using formula (22), we obtain

$$\left[s'\left(\dot{R}^4+R^3\ddot{R}\right)-k_sR^3\phi_1\left(\boldsymbol{e}\mathbf{v}\right)+k_sR^2\dot{R}G\left(\boldsymbol{g}\mathbf{v}\right)\right]\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right)=0.$$
(23)

Obviously for the condition (23) to fulfill, it is enough to equate to zero multiplier. As the vector lies in the normal plane of a directrix, then

$$\boldsymbol{e}_0=\left(\boldsymbol{e}_0\mathbf{v}\right)\mathbf{v}+\left(\boldsymbol{e}_0\boldsymbol{\beta}\right)\boldsymbol{\beta};\ \left(\boldsymbol{g}_0\boldsymbol{\beta}\right)=-\left(\boldsymbol{e}_0\mathbf{v}\right);\quad \left(\boldsymbol{g}_0\mathbf{v}\right)=\left(\boldsymbol{e}_0\boldsymbol{\beta}\right);$$
$$\left(\boldsymbol{e}_0\mathbf{v}\right)^2+\left(\boldsymbol{e}_0\boldsymbol{\beta}\right)^2=1;\quad \left(\boldsymbol{e}_0\boldsymbol{\beta}\right)'=-\left(\boldsymbol{e}_0\mathbf{v}\right)'\frac{\left(\boldsymbol{e}_0\mathbf{v}\right)}{\left(\boldsymbol{e}_0\boldsymbol{\beta}\right)}.$$

Considering these ratios, we have

$$\left(\boldsymbol{e}_0'\boldsymbol{g}_0\right)=\left[\left(\boldsymbol{e}_0\mathbf{v}\right)\mathbf{v}+\left(\boldsymbol{e}_0\boldsymbol{\beta}\right)\boldsymbol{\beta}\right]'\boldsymbol{g}_0=\left[\left(\boldsymbol{e}_0\mathbf{v}\right)'\mathbf{v}+\left(\boldsymbol{e}_0\boldsymbol{\beta}\right)'\boldsymbol{\beta}\right]\boldsymbol{g}_0+\left[\left(\boldsymbol{e}_0\mathbf{v}\right)\mathbf{v}'+\left(\boldsymbol{e}_0\boldsymbol{\beta}\right)\boldsymbol{\beta}'\right]\boldsymbol{g}_0=$$
$$=\left(\boldsymbol{e}_0\mathbf{v}\right)'\left[\mathbf{v}-\frac{\left(\boldsymbol{e}_0\mathbf{v}\right)}{\left(\boldsymbol{e}_0\boldsymbol{\beta}\right)}\boldsymbol{\beta}\right]\boldsymbol{g}_0+\chi_s\left[\left(\boldsymbol{e}_0\mathbf{v}\right)\boldsymbol{\beta}-\left(\boldsymbol{e}_0\boldsymbol{\beta}\right)\mathbf{v}\right]\boldsymbol{g}_0=\frac{\left(\boldsymbol{e}_0\mathbf{v}\right)'}{\sqrt{1-\left(\boldsymbol{e}_0\mathbf{v}\right)^2}}+\chi_s=0$$
.

By integrating the last ratio, we obtain

$$(\boldsymbol{e}_0\mathbf{v}) = \cos\theta, \quad (24)$$

$\theta(u) = -\int \chi_s du + \theta_0$, θ_0 is the initial angle (u = 0) between vector $\bar{e}_0$ and normal vector $\bar{\nu}$ of a directrix (an integration constant).

Thus, at movement of a flat generatrix along spatial directrix of Monge's ruled surface the angle between a normal of a directrix and coordinate system of a generatrix changes. In case of a flat directrix ($\chi_s = 0$) this angle remains constant $\theta = \theta_0$.

As result we obtain:

$$\boldsymbol{n} = \boldsymbol{\tau}; \quad \boldsymbol{n}' = k_s \mathbf{v}; \quad \boldsymbol{n}'' = -k_s^2 \boldsymbol{\tau} + k_s' \mathbf{v} + k_s \chi_s \boldsymbol{\beta};$$

$$(\boldsymbol{e\tau}) = (\boldsymbol{g\tau}) = 0;$$

$$\boldsymbol{e}_0' = -(\boldsymbol{e}_0 \boldsymbol{n}')\boldsymbol{n} = -k_s(\boldsymbol{e}_0\mathbf{v})\boldsymbol{\tau}; \quad (\boldsymbol{e}_0'' \boldsymbol{g}_0) = -k_s^2 (\boldsymbol{e}_0 \mathbf{v})(\boldsymbol{g}_0 \mathbf{v});$$

$$(\boldsymbol{e}\mathbf{v}) = (\boldsymbol{e}_0\mathbf{v})\sin v + (\boldsymbol{g}_0\mathbf{v})\sin v = \cos\theta\cos v - \sin\theta \sin v = \cos\omega;$$

$$(\boldsymbol{e}\bar{\beta}) = \sin\omega; \quad (\boldsymbol{g}\mathbf{v}) = -\sin\omega; \quad (\boldsymbol{g\beta}) = \cos\omega;$$

$$\omega = \upsilon + \theta.$$

And taking into account formulas (21)

$$T_{11} = T_{12} = 0; \; T_{13} = s' - R k_s \cdot \cos\omega; \; T_{21} = \dot{R};$$

$$T_{22} = R; \quad T_{23} = 0;$$

$$T_{31} = T_{13} k_s \cos\omega; T_{32} = -T_{13} k_s \sin\omega;$$

$$T_{33} = s'' - R k_s' \cos\omega - R k_s \chi_s \sin\omega;$$

$$T_{41} = T_{42} = 0; \; T_{43} = -k_s [R\cos\omega]^{\cdot}; \; T_{51} = \ddot{R} - R;$$

$$T_{52} = 2\dot{R}; \quad (25)$$

Coefficients of quadratic forms:

$$A = \sqrt{E} = s' - k_s R \cos\omega; \quad B = \sqrt{G} = \sqrt{R^2 + \dot{R}^2} = s_0';$$

$$F = 0; \quad \Sigma = AB;$$

$$\overline{m} = \frac{-R\boldsymbol{e} + \dot{R}\boldsymbol{g}}{B}; \; L = -[R\cos\omega]^{\cdot} k_s \frac{A}{B}; \; M = 0;$$

$$N = \frac{(R - \ddot{R})R + 2\dot{R}^2}{B}. \quad (26)$$

The main curvatures:

$$k_1 = \frac{R\cos\omega + \dot{R}\sin\omega}{AB} k_s;$$

$$k_2 = \frac{R(R - \ddot{R}) + 2\dot{R}^2}{B^3} = k_0. \quad (27)$$

s_0', k_0 are the differential and the curvature of a generatrix.

The fact that curvature k_2 of a carved surface is equal to curvature of a generatrix, shows that generatrices of Monge's ruled surfaces are geodesic lines of this surface.

If the flat directrix is a circle, then the ruled surface is a surface of revolution. Thus we obtain: $s' = a$; $k = 1/a$; $k_s = 1$; $\theta = \theta_0$. Taking into account these ratios, the coefficients of quadratic forms and the main curvature are calculated by formulas (26), (27).

If the radius of the directing circle tends to zero ($s' = 0$), then we will obtain a surface of revolution in the spherical system of coordinates.

If the equation of the flat generatrix of the ruled surface is defined in a parametrical form in the rectangular system of coordinates:

$$X = X(\upsilon); \quad Y = Y(\upsilon), \quad (28)$$

then the equation of a ruled surface will be written as follows:

$$\boldsymbol{\rho}(u,v) = \boldsymbol{r}(u) + X\,\boldsymbol{e}_0(u,\theta) + Y(v)\boldsymbol{g}_0(u,\theta), \quad (29)$$

$\boldsymbol{e}_0(u,\theta) = \mathbf{v}\cos\theta + \boldsymbol{\beta}\sin\theta$, $\boldsymbol{g}_0(u,\theta) = -\mathbf{v}\sin\theta + \boldsymbol{\beta}\cos\theta$ is unitary vectors of the rectangular system of coordinates in the normal plane of the directrix - $\boldsymbol{r}(u)$; $\theta = \theta(u) = -\int \chi_s du + \theta_0$ is the between unitary vectors $\boldsymbol{e}_0$, $\boldsymbol{g}_0$ and normal $\mathbf{v}$ and a binormal vectors $\boldsymbol{\beta}$ of a spatial directrix. For a flat directrix it is accepted $\theta = \theta_0 = const$, particularly $\theta_0 = 0$.

Examples of ruled surfaces are plotted using MathCAD system are given in fig. 2. The equations of surfaces in curvature lines were used.

Ruled surfaces with a directing ellipse $x = 2{,}5\cos u$; $y = \sin u$ and generatrices: (fig. 2, a) a □ parabola $z = 0{,}1v^2$ ($0 \le v \le 1$); (fig. 2,b) a sinusoid $\tilde{z} = 0{,}5\sin(kv)$ ($0 \le v \le \pi$, p = 6). with an axis turned to the ellipse plane for 30°.

Fig. 2c, the directrix is a parabola $x = u$, $y = u^2$ ($-1 \le u \le 1$), the generatrix is a sinusoid $z = 2\sin(pv)$ ($0 \le v \le \pi$, p = 4). Fig. 2*d* – the directrix is an involute of a circle $x = 0{,}5(\cos u + u\sin u)$, $y = 0{,}5(\sin u - u\cos u)$ ($\pi/2 \le u \le 4\pi$,), the generatrix is a hyperbolic curve $x = 0{,}5chv$; $y = 0{,}8shv$ ($-1{,}5 \le v \le 1{,}5$).

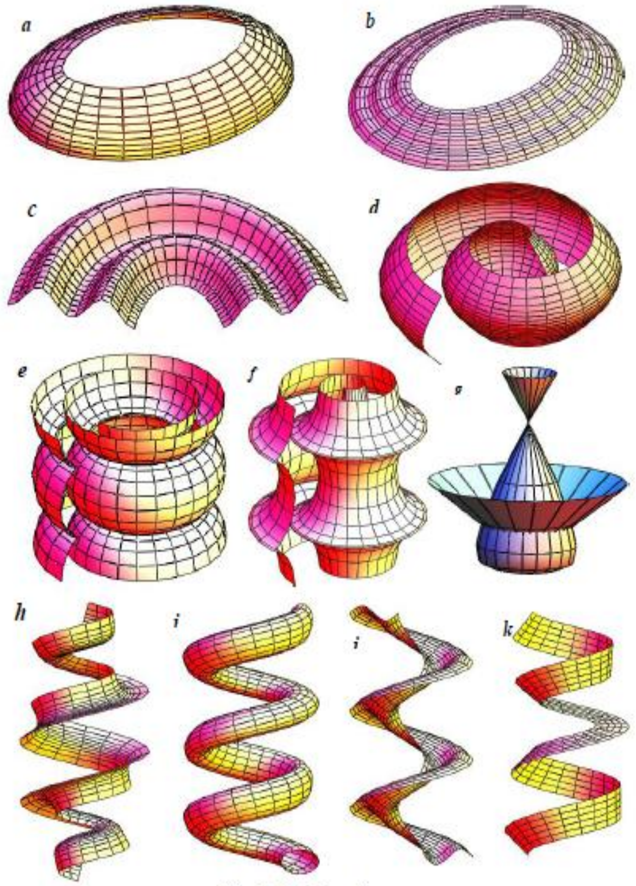

Fig. 2. Ruled surfaces

Fig.2,*e*,*f* – the directrix is conic flat spiral

$x = 5e^{0,08u}\cos u$, $x = 5e^{0,08u}\cos u$ $(0 \le u \le 4\pi)$, the generatrix is cycloid $\tilde{x} = a(v - \sin v)$, $\tilde{y} = a(1 - \cos v)$: *∂*) $a = -2$, *e*) $a = -2$, $(-\pi \le v \le 3\pi)$. Fig. 2,*g* – the directrix is a straight line (*z-axe*) , the generatrix is a parabola $z = v^2$ with a tilt angle to directing equal to 30^o (a surface of revolution).

The ruled surfaces with the screw directrix $x = 8\cos u$ $y = 8\sin u$, $z = 2u$, and a directrix: a parabola (fig. 2, *h*), an ellipse – (fig. 2, *i*), a sinusoid (fig. 2, *j*) and a straight line (fig. 2, *k*).

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