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Instrumented System for the Solution of Static Problems on the Theory of Elasticity for a Multilayer Elastic Foundation

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Abstract- ***The article presents an instrumented system developed by the author on the basis of analytical methods. The essence of analytical methods is given in the text. The compute kernel of the instrumented system is represented by Maxima computer mathematics. Examples of instrumented system operation constitute the fully automated development of analytical solutions of static problems on the theory of elasticity for a multilayer elastic foundation in two-dimensional and three-dimensional setting.***

Key words: computer mathematics system, instrumented system, preprocessor, theory of elasticity.

I. INTRODUCTION

The developed instrumented system uses a new approach to the automation of solution of elasticity theory problem for an elastic foundation in two-dimensional and three-dimensional setting. It is based on a combination of analytical and numerical methods. The method of general solution development with the help of simplifying symbols is used as the main analytical method for solution of elasticity theory problems. Further, symbolic integration methods are used – Fourier integrals, and then numerical integration methods – Simpson method, method of trapezoids, rectangles – for problems in two-dimensional setting, method of cells – for three-dimensional setting.

II. MATHEMATICAL THEORY

A. Two-Dimensional Case

We are considering a layered elastic foundation composed of a number of horizontal layers with different elasticity characteristics. Thickness of the whole foundation is h, thickness and elastic constants of layers – h_m, ν_m, G_m. m – layer number. An assumption is made – in the transition through the layer contact plane, motion and stress vectors vary in continuous manner. For the purpose of reducing notations, new notations are introduced for the sought quantities of motion and stress u, υ, τ_{xy}, σ_y:

$$\left.\begin{array}{l} G u(x, y) = U_1, \quad G \upsilon(x, y) = U_2, \\ \tau_{xy}(x, y) = U_3, \quad \sigma_y(x, y) = U_4, \quad \sigma_x(x, y) = U_6. \end{array}\right\} \quad (1)$$

Initial functions

$$u_0(x,\ y) = U_1^0,\ \upsilon_0(x,\ y) = U_2^0,$$

$$\tau_{xy}^0(x,\ y) = X_0(x,\ y) = U_4^0,\ \sigma_y^0(x,\ y) = Y_0(x,\ y) = U_5^0.$$

Motion and stress in the first elastic foundation layer operating under conditions of plane-deformable sate, can be presented as follows:

$$U_i = \sum_{k=1}^{4} \int_{-\infty}^{+\infty} \left(A_{ik}(y, \alpha)\ f_i(x, \alpha)\ u_k^0(\alpha) + B_{ik}(y, \alpha)\ g_i(x, \alpha)\ u_k^{0^*} \right) d\alpha\ , \ (i = \overline{1,5}) \quad (2)$$

where $f_i(x, \alpha) = \sin \alpha x$ for $i = 1, 4$ and $f_i(x, \alpha) = \cos \alpha x$ for $i = 2, 3, 5$;

$g_i(x, \alpha) = \cos \alpha x$ for $i = 1, 4$ and $g_i(x, \alpha) = \sin \alpha x$ for $i = 2, 3, 5$;

A_{ik} denotes the known functions, numerical form of operators L according to operation [6] for initial functions

$$\left.\begin{array}{l} U_1^0 = \sum\limits_{n=1}^{\infty} u_{1n}^0 \sin \alpha_n x, \quad U_2^0 = \sum\limits_{n=1}^{\infty} u_{2n}^0 \cos \alpha_n x, \\ U_3^0 = \sum\limits_{n=1}^{\infty} u_{3n}^0 \sin \alpha_n x, \quad U_4^0 = \sum\limits_{n=1}^{\infty} u_{4n}^0 \cos \alpha_n x, \end{array}\right\};$$

$$A_{11} = \frac{1}{G}\left(ch\ \alpha y + \frac{\alpha\, y\ sh\ \alpha y}{2(1-\nu_1)} \right),\ \ A_{23} = \frac{1}{4G\ (1-\nu_1)}\left(\frac{3-4\nu_1}{\alpha} sh\ \alpha\, y - y\, ch\ \alpha \right)$$

etc.;

functions B_{ik} are determined the same way as functions A_{ik} from expressions of operation [6], but for initial functions:

$$\left.\begin{array}{l} U_1^0 = \sum\limits_{n=1}^{\infty} u_{1n}^0 \cos \alpha_n x, \quad U_2^0 = \sum\limits_{n=1}^{\infty} u_{2n}^0 \sin \alpha_n x, \\ U_3^0 = \sum\limits_{n=1}^{\infty} u_{3n}^0 \cos \alpha_n x, \quad U_4^0 = \sum\limits_{n=1}^{\infty} u_{4n}^0 \sin \alpha_n x, \end{array}\right\}$$

i.e. it differs from Ribiere formula, where there first goes sin and then cos.

Stress and motion in a random m layer of unbounded foundation. In case of continuity of motion and stress vectors in the transition through layer contact plane, they are determined by the formulae:

$$U_i = \sum_{k=1}^{4} \int_{-\infty}^{+\infty} \left(A_{ik}^*\ f_i\, u_k^0 + B_{ik}^*\ g_i\, u_k^{0^*} \right) d\alpha\ ,\ \ (i = \overline{1,5}) \quad (3)$$

here, matrices $\left\| A_{ik}^* \right\|$ and $\left\| B_{ik}^* \right\|$ represent matrix product respectively

$$\left\| A_{ik}^{(j)}(\alpha, h_j, \nu_j, G_j) \right\| \qquad \left\| A_{ik}^{(m)}(\alpha, y, \nu_m, G_m) \right\|$$

and

$$\left\| B_{ik}^{(j)}(\alpha, h_j, \nu_j, G_j) \right\| \qquad \left\| B_{ik}^{(m)}(\alpha, y, \nu_m, G_m) \right\|$$

$$\left(j = 1, 2, \ldots, m-1 \right),\ \ \left(h_{m-1} \le y \le h_m \right).$$

Unknown functions u_k^0 and $u_k^{0^*}$ are determined from boundary conditions on the plane.

B. Three-Dimensional Case

We are considering a layered elastic foundation composed of a number of horizontal layers with different elasticity characteristics.

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Thickness of the whole foundation is h, thickness and elastic constants of layers – h_m, ν_m, G_m. m – layer number. An assumption is made – in the transition through the layer contact plane, motion and stress vectors vary in continuous manner. For the purpose of reducing notations, new notations are introduced for the sought quantities of motion and stress u, υ, w, τ_{xz}, τ_{yz}, τ_{xy}, σ_z, σ_x, σ_y:

$$\left.\begin{array}{l} G u(x,\ y)=U_1,\quad G\upsilon(x,\ y)=U_2,\ \ G w(x,\ y)=U_3,\\ \tau_{xz}(x,\ y)=U_4,\quad \tau_{yz}(x,\ y)=U_5,\ \sigma_z(x,\ y)=U_6,\\ \sigma_x(x,\ y)=U_7,\quad \sigma_y(x,\ y)=U_8,\ \tau_{xy}(x,\ y)=U_9. \end{array}\right\} \qquad (4)$$

Initial functions

$$\begin{array}{l} u_0(x,\ y)=U_1^0,\quad \upsilon_0(x,\ y)=U_2^0,\ \ w_0(x,\ y)=U_3^0,\\ \tau_{xz}^0(x,\ y)=X_0(x,\ y)=U_4^0,\ \tau_{yz}^0(x,\ y)=Y_0(x,\ y)=U_5^0,\\ \sigma_z^0(x,\ y)=Z_0(x,\ y)=U_6^0,\ \ \sigma_x^0(x,\ y)=U_7^0,\\ \sigma_y^0(x,\ y)=U_8^0,\ \ \tau_{xy}^0(x,\ y)=U_9^0. \end{array}$$

Motion and stress in the first elastic foundation layer operating under conditions of stress sate, can be presented as follows:

$$\begin{array}{l} U_i=\sum\limits_{k=1}^{4}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\Big(A_{ik}(z,\alpha,\beta)\ f_i(x,\ y,\alpha,\beta)\ u_k^0(\alpha,\beta)+\\ B_{ik}(z,\alpha,\beta)\ g_i(x,\ y,\alpha,\beta)\ u_k^{0^*}+\\ C_{ik}(z,\alpha,\beta)\ \upsilon_i(x,\ y,\alpha,\beta)\ u_k^{0^{**}}(\alpha,\beta)+ \qquad (i=\overline{1,9})\\ D_{ik}(z,\alpha,\beta)\ \omega_i(x,\ y,\alpha,\beta)\ u_k^{0^{***}}(\alpha,\beta)\ d\alpha\, d\beta\ , \end{array} \qquad (5)$$

where $f_i(x,\ y,\ \alpha,\ \beta)=\sin\alpha x\cos\beta y$ for $i=1,\ 4,\ 7$,

$f_i(x,\ y,\ \alpha,\ \beta)=\cos\alpha x\sin\beta y$ for $i=2,\ 5,\ 8$,

$f_i(x,\ y,\ \alpha,\ \beta)=\cos\alpha x\cos\beta y$ for $i=3,\ 6,\ 9$;

$g_i(x,\ y,\ \alpha,\ \beta)=\cos\alpha x\sin\beta y$ for $i=1,\ 4,\ 7$,

$g_i(x,\ y,\ \alpha,\ \beta)=\sin\alpha x\cos\beta y$ for $i=2,\ 5,\ 8$,

$g_i(x,\ y,\ \alpha,\ \beta)=\sin\alpha x\sin\beta y$ for $i=3,\ 6,\ 9$;

$\upsilon_i(x,\ y,\ \alpha,\ \beta)=\sin\alpha x\sin\beta y$ for $i=1,\ 4,\ 7$,

$\upsilon_i(x,\ \alpha)=\cos\alpha x\cos\beta y$ for $i=2,\ 5,\ 8$,

$\upsilon_i(x,\ y,\ \alpha,\ \beta)=\cos\alpha x\sin\beta y$ for $i=3,\ 6,\ 9$;

$\omega_i(x,\ y,\ \alpha,\ \beta)=\cos\alpha x\cos\beta y$ for $i=1,\ 4,\ 7$,

$\omega_i(x,\ \alpha)=\sin\alpha x\sin\beta y$ for $i=2,\ 5,\ 8$,

$\omega_i(x,\ y,\ \alpha,\ \beta)=\sin\alpha x\cos\beta y$ for $i=3,\ 6,\ 9$;

A_{ik} denotes the known functions, numerical form of operators L according to operation [6] for initial functions

$$\left.\begin{array}{ll} U_1^0=\sum\limits_{n=1}^{\infty}u_{1nm}^0\sin\alpha_n x\cos\beta_k y, & U_2^0=\sum\limits_{n=1}^{\infty}u_{2nm}^0\cos\alpha_n x\sin\beta_k y,\\ U_3^0=\sum\limits_{n=1}^{\infty}u_{3nm}^0\cos\alpha_n x\cos\beta_k y, & U_4^0=\sum\limits_{n=1}^{\infty}u_{4nm}^0\sin\alpha_n x\cos\beta_k y,\\ U_5^0=\sum\limits_{n=1}^{\infty}u_{5nm}^0\cos\alpha_n x\sin\beta_k y, & U_6^0=\sum\limits_{n=1}^{\infty}u_{6nm}^0\cos\alpha_n x\cos\beta_k y, \end{array}\right\}$$

functions B_{ik} are determined the same way as functions A_{ik} from expressions of operation [6], but for initial functions:

$$\left.\begin{array}{ll} U_1^0=\sum\limits_{n=1}^{\infty}u_{1nm}^0\cos\alpha_n x\sin\beta_k y, & U_2^0=\sum\limits_{n=1}^{\infty}u_{2nm}^0\sin\alpha_n x\cos\beta_k y,\\ U_3^0=\sum\limits_{n=1}^{\infty}u_{3nm}^0\sin\alpha_n x\sin\beta_k y, & U_4^0=\sum\limits_{n=1}^{\infty}u_{4nm}^0\cos\alpha_n x\sin\beta_k y,\\ U_5^0=\sum\limits_{n=1}^{\infty}u_{5nm}^0\sin\alpha_n x\cos\beta_k y, & U_6^0=\sum\limits_{n=1}^{\infty}u_{6nm}^0\sin\alpha_n x\sin\beta_k y. \end{array}\right\}$$

functions C_{ik} for

$$\left.\begin{array}{ll} U_1^0=\sum\limits_{n=1}^{\infty}u_{1nm}^0\sin\alpha_n x\sin\beta_k y, & U_2^0=\sum\limits_{n=1}^{\infty}u_{2nm}^0\cos\alpha_n x\cos\beta_k y,\\ U_3^0=\sum\limits_{n=1}^{\infty}u_{3nm}^0\cos\alpha_n x\sin\beta_k y, & U_4^0=\sum\limits_{n=1}^{\infty}u_{4nm}^0\sin\alpha_n x\sin\beta_k y,\\ U_5^0=\sum\limits_{n=1}^{\infty}u_{5nm}^0\cos\alpha_n x\cos\beta_k y, & U_6^0=\sum\limits_{n=1}^{\infty}u_{6nm}^0\cos\alpha_n x\sin\beta_k y, \end{array}\right\}$$

functions D_{ik} for

$$\left.\begin{array}{ll} U_1^0=\sum\limits_{n=1}^{\infty}u_{1nm}^0\cos\alpha_n x\cos\beta_k y, & U_2^0=\sum\limits_{n=1}^{\infty}u_{2nm}^0\sin\alpha_n x\sin\beta_k y,\\ U_3^0=\sum\limits_{n=1}^{\infty}u_{3nm}^0\sin\alpha_n x\cos\beta_k y, & U_4^0=\sum\limits_{n=1}^{\infty}u_{4nm}^0\cos\alpha_n x\cos\beta_k y,\\ U_5^0=\sum\limits_{n=1}^{\infty}u_{5nm}^0\sin\alpha_n x\sin\beta_k y, & U_6^0=\sum\limits_{n=1}^{\infty}u_{6nm}^0\sin\alpha_n x\cos\beta_k y. \end{array}\right\}$$

Stress and motion in a random m layer of unbounded foundation. In case of continuity of motion and stress vectors in the transition through layer contact plane, they are determined by the formulae:

$$U_i=\sum_{k=1}^{4}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\Big(A_{ik}^*\,f_i\,u_k^0+B_{ik}^*g_i\,u_k^{0^*}+C_{ik}^*\,\upsilon_i\,u_k^{0^{**}}+D_{ik}^*\,\omega_i\,u_k^{0^{***}}\Big)\,d\alpha\,d\beta\ , \qquad (7)$$

here, matrices $\left\|A_{ik}^*\right\|$ and $\left\|B_{ik}^*\right\|$ represent matrix product respectively

$$\left\|A_{ik}^{(j)}(\alpha,h_j,\nu_j,G_j)\right\| \qquad \left\|A_{ik}^{(m)}(\alpha,y,\nu_m,G_m)\right\|$$

and

$$\left\|B_{ik}^{(j)}(\alpha,h_j,\nu_j,G_j)\right\| \qquad \left\|B_{ik}^{(m)}(\alpha,y,\nu_m,G_m)\right\|$$

$$(j=1,\ 2,\ \ldots,\ m-1),\ \ (h_{m-1}\le y\le h_m).$$

Unknown functions u_k^0, $u_k^{0^*}$, $u_k^{0^{**}}$, $u_k^{0^{***}}$ are found from boundary conditions on a semispace.

III. EXAMPLES

The developed instrumented system performs operations of simplification and transfer of differential operators from symbolic presentation into the form of numerical series, and contracts the results being obtained. Rules of result processing depend on the parity of operators and presence of multiplication and division operations in the symbolic representation [6]. The system works with two-dimensional and tree-dimensional equations of the elasticity theory.

A series of prepared operations for preprocessor are given in the following sequence of commands in the language of Maxima computer mathematics system:

```
n_sloy:2$
U[0,2]:0$ U1[0,2]:U[0,2]$
U[0,3]:0$ U1[0,3]:U[0,3]$
Ph[3]:0$
```

Ph[4]:1/2/%pi*p(lambda)*cos(alpha*(lambda-x))$

where n_sloy:2$ - is setting the number of layers.

List

U[0,2]:0$ U1[0,2]:U[0,2]$
U[0,3]:0$ U1[0,3]:U[0,3]$

initial functions of the problem [7].

A. Two-Dimensional Problem

This is a problem on the equilibrium of two-layer elastic foundation under the impact of vertical evenly distributed load *p* applied along the top straight line of the foundation *y* = *h*. Foundation length is $|A_1 - A_2|$, width is *h*. The segment under the load *p* $[a_1, a_2]$, at $A_1 < a_1 < a_2 < A_2$. An assumption is made that the foundation is located on hard land, and there is no friction with the land.
For problem solution, a coordinate system shall be built, as it is shown in Figure 1.

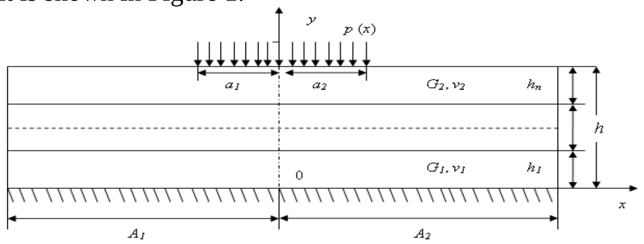


Figure 1: Multilayer elastic foundation, two-dimensional case

On the initial line $y = 0$ motion U_2 and tangent stress equal 0, in formula (2) we have to set $u_2^0 = u_2^{0^*} = u_4^0 = u_4^{0^*} = 0$. For determination of other unknown functions, boundary problem conditions on the top line *y* = *h* are used. Load *p* is presented in the form of Fourier integral:

$$p\,(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\alpha \int_{a_1}^{a_2} p\,(\lambda)\,\cos\alpha(\lambda - x)\,d\lambda \cdot \qquad (8)$$

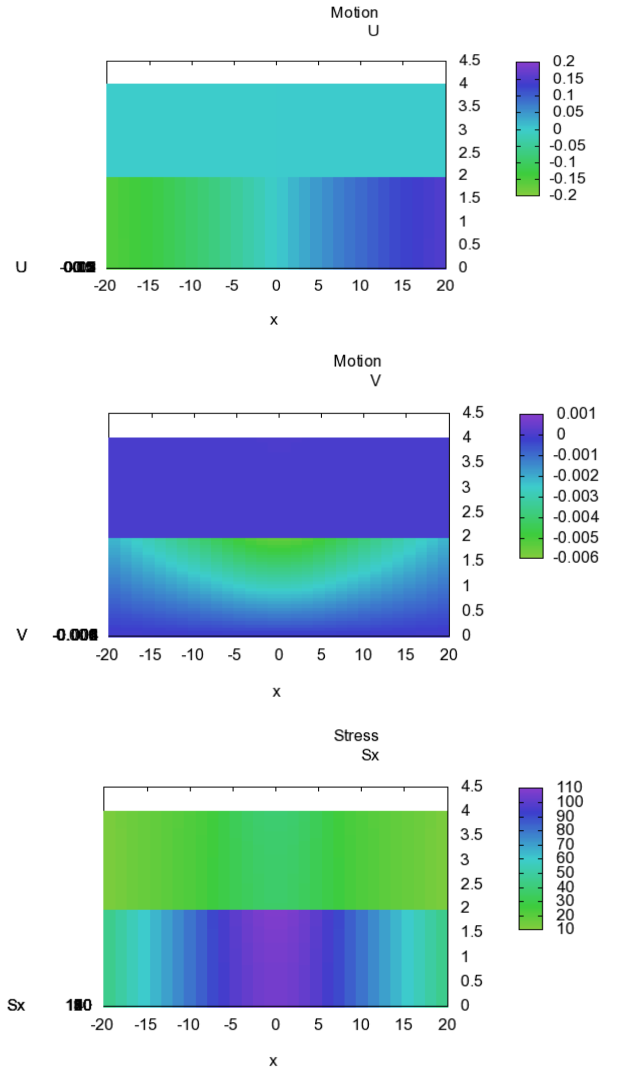
We set the expressions for stress U_3 and U_4 from the boundary line *y* = *h* respectively equal to value (8) and 0, and we get a system of linear algebraic equations for determination of unknown initial functions

$$\sum_{i=1,3} A_{41}^*(\alpha h) u_i^0 = -\frac{1}{2\pi}\int_{a_1}^{a_2} p(\lambda)\cos\alpha\lambda\, d\lambda;\ \sum_{i=1,3} A_{31}^*(\alpha h) u_i^0 = 0;$$
$$\sum_{i=1,3} B_{41}^*(\alpha h) u_i^0 = -\frac{1}{2\pi}\int_{a_1}^{a_2} p(\lambda)\sin\alpha\lambda\, d\lambda;\ \sum_{i=1,3} B_{31}^*(\alpha h) u_i^0 = 0. \qquad (9)$$

When evaluating from (9) functions u_i^0 and $u_i^{0^*}$ $(i = 1, 3)$ and presenting them in the form of (2), we find the stressed and deformed state of the semiplane. Expressions (2) are not integrable in primitive functions, so methods of numerical integration with finite bounds are used for building numerical solutions. Foundation boundaries A_1 and A_2 are set as finite as a computer operates a finite set of data.
Initial data for computation:

$A1 = -20$
$A2 = 20$
$a1 = -5$
$a2 = 5$
$h_0 = 0$
$h_1 = 2$
$h_2 = 2$
$E = 10000.0$
$G_1 = \frac{5000.0}{\nu_1 + 1}$
$G_2 = \frac{50000.0}{\nu_1 + 1}$
$\nu_1 = 0.2$
$\nu_2 = 0.3$
$p(lambda) = -1$.

Graphs of numerical solutions in the form of density functions are shown in Figure 2.



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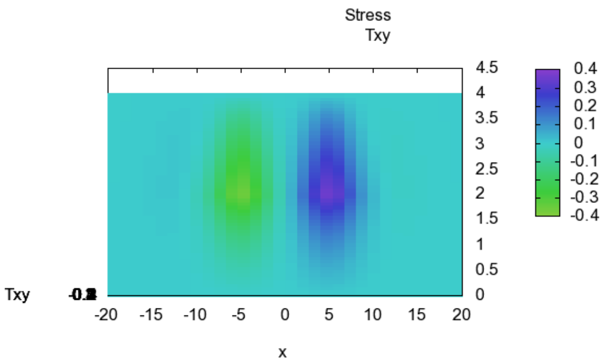


Figure 2: Results of solution of elasticity theory two-dimensional problem

B. Three-Dimensional Problem

This is a problem on the equilibrium of two-layer elastic foundation under the impact of vertical evenly distributed load *p* applied along the top plane of the foundation *z* = *h*. Foundation length is $|A_1 - A_2|$, width is $|B_1 - B_2|$, height is *h*. The segments under the load *p* $[a_1, a_2]$ and $[b_1, b_2]$, at $A_1 < a_1 < a_2 < A_2$ and $B_1 < b_1 < b_2 < B_2$. An assumption is made that the foundation is located on hard land, and there is no friction with the land.

For problem solution, a coordinate system shall be built, as it is shown in Figure 3.

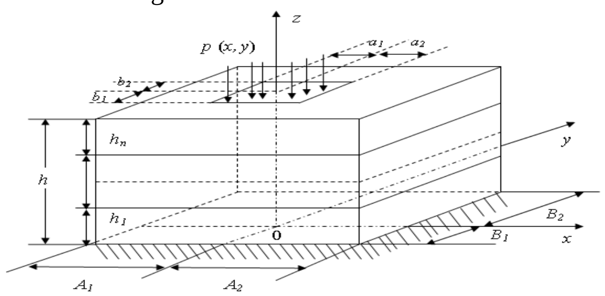


Figure 3: Multilayer elastic foundation, three-dimensional case

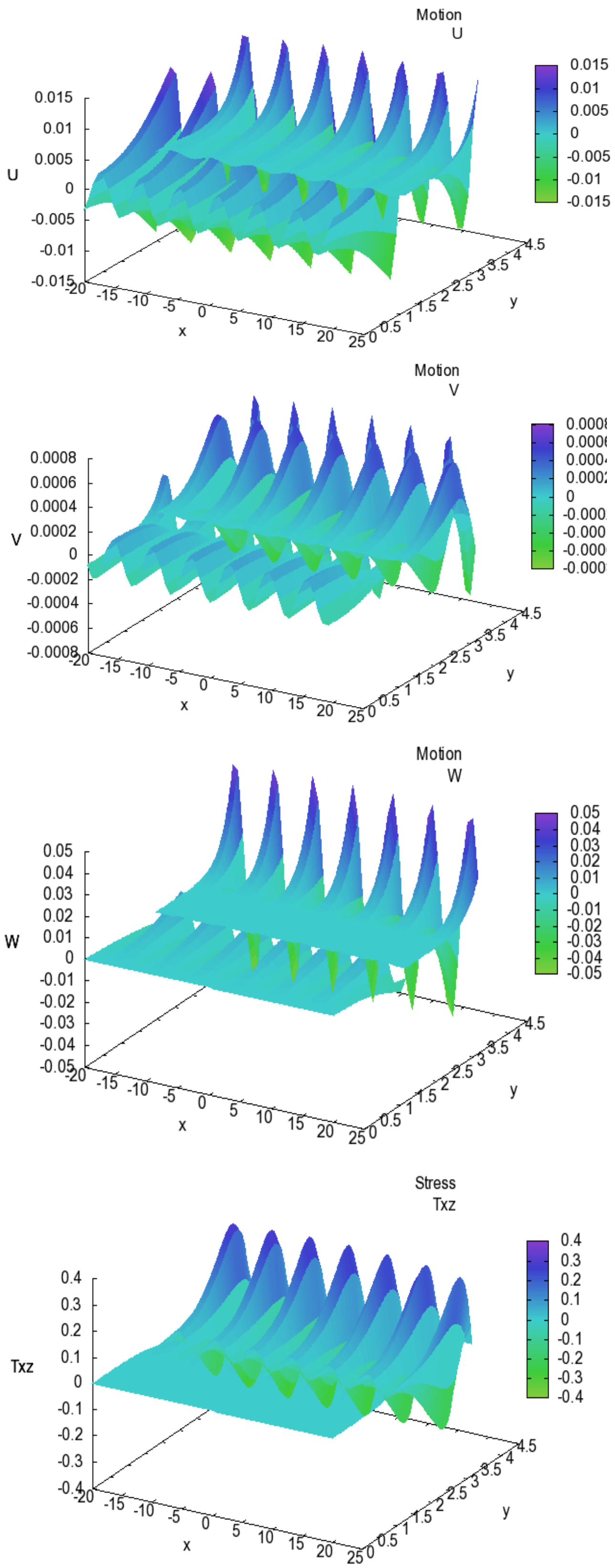
For the initial plane $z = 0$ motion U_3 and tangent stress U_4, U_5 equal 0, so in formula (2) we have to set $u_3^0 = u_3^{0^*} = u_3^{0^{**}} = u_3^{0^{***}} = 0$, $u_4^0 = u_4^{0^*} = u_4^{0^{**}} = u_4^{0^{***}} = 0$ and $u_5^0 = u_5^{0^*} = u_5^{0^{**}} = u_5^{0^{***}} = 0$. For determination of other unknown functions, boundary problem conditions on the top plane *z* = *h* are used. In the three-dimensional case, double integration is used; we take double Fourier integral of the load $p(\lambda, \eta)$:

$$p\,(x,\,y) = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\alpha\, d\beta \int_{a_1}^{a_2}\int_{b_1}^{b_2} p\,(\lambda,\,\eta)\cos\alpha(\lambda - x)\cos\beta(\eta - y)\,d\lambda\, d\eta \cdot$$

Further, with the help of the scheme developed by the author, equations are formed, wherefrom initial functions $u_i^0, u_i^{0^*}, u_i^{0^{**}}, u_i^{0^{***}}$ $(i = 1, 2, 6)$ are found for zero u_i^0, $u_i^{0^*}, u_i^{0^{**}}, u_i^{0^{***}}$ $(i = 3, 4, 5)$. On a computer, the computation is performed for finite bounds, because taking double integral of the expression (7) is a complicated task (integrals are not presented in known functions and can be calculated only in numerical terms). Such bounds are the variables A_1 and A_2, B_1 and B_2.

Initial data for three-dimensional problem computation:

General view of graphs for numerical three-dimensional problem solution is shown in Figure 4.



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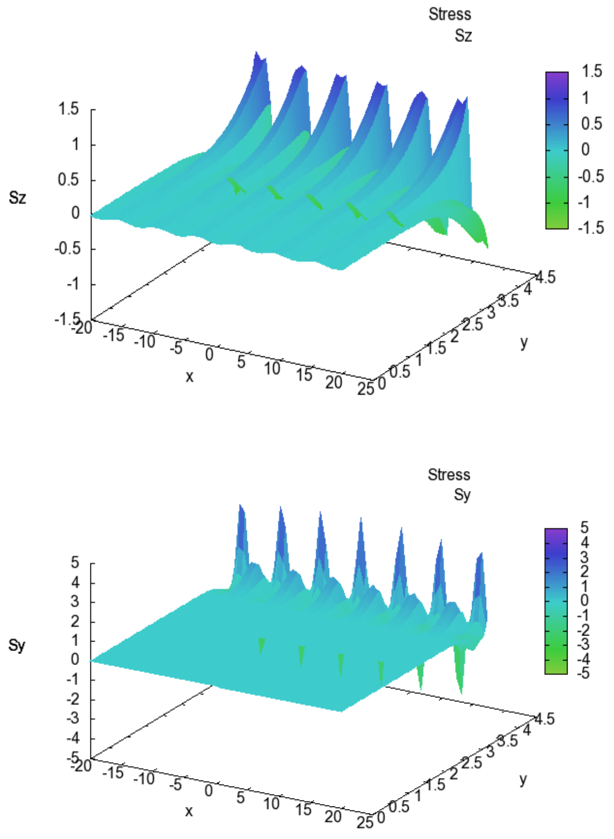

Figure 4: Results of solution of elasticity theory three-dimensional problem

IV. CONCLUSION

The work describes the developed instrumented system of static problem solution in two-dimensional and three-dimensional setting for an elastic multilayer foundation. It presents the main analytical methods used in the system for development of the set problem solutions. With the help of the developed instrumented system, it is possible to solve more complicated elasticity theory problems, analytical solutions whereof could not be previously obtained through analytical means by researchers. Software implementations of new algorithms for analytical solution development allow us using computers in new areas of mathematical modeling, where determinations of complex mathematical formulae are used. The instrumented system allows shifting the process of determining mathematical formulae of solutions on to a computer. It constitutes the software implementation of algorithms for building analytical solutions of elasticity theory static problems.

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