Bayesian Estimates based on Combined Hybrid Censored Data from the Modified Weibull Lifetime Model

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***Abstract*— In this article, we derive Bayesian stimations of three parameters and some survival time parameters e.g. reliability and hazard functions in the modified Weibull distribution based on combined hybrid censored data. Finally, a real life data set and simulation data are used to illustrate the discussed methodology.**

***Index Terms*— Bayesian estimation; modified Weibull distribution; order statistics; combined hybrid censored data.**

I. INTRODUCTION

Type-I and Type-II censoring schemes are the two most popular censoring schemes which are used in practice. The mixture of Type-I and Type-II censoring schemes has been discussed in the literature for this purpose, is known as the hybrid censoring scheme which was first introduced by Epstein (1954). But recently it becomes quite popular in the reliability and life-testing experiments, see for example Fairbanks et al. (1982), Draper and Guttman (1987), Chen and Bhattacharya (1988), Jeong et al.(1996), Childs et al. (2003) and Gupta and Kundu (2006). Huang and Yang (2010) considered a combined hybrid censoring sampling (HCS) scheme which define as follows. Fix $k, r \in \{1,2,\ldots,n\}$ and $T_1, T_2 \in (0,\infty)$ such $k < r$ and $T_1 < T_2$. Let T^* denote the terminating time of the experiment. If the k th failure occurs before time T_1, the experiment terminates at $\min\{X_{r:n}, T_1\}$. And if the k th failure occurs between T_1 and T_2, the experiment is terminated at $X_{k:n}$ and finally if the k th failure occurs after time T_2, then the experiment terminates at T_2. For our later convenience, we abbreviate this scheme as combined HCS $(k, r; T_1, T_2)$. In fact, this system contains the following six cases, and obviously, in each case some part of data are unobservable,

(1) For $0 < T_1 < X_{k:n} \quad (< T_2 < X_{r:n}), \quad T^* = X_{k:n}$,

(2) For $0 < T_1 < X_{k:n} \quad (< X_{r:n} < T_2), \quad T^* = X_{k:n}$,

(3) For $0 < T_1 < T_2 \quad (< X_{k:n} < X_{r:n}), \quad T^* = T_2$,

(4) For $0 < X_{k:n} < X_{r:n} \quad (< T_1 < T_2), \quad T^* = X_{r:n}$,

(5) For $0 < X_{k:n} < T_1 \quad (< X_{r:n} < T_2), \quad T^* = T_1$,

(6) For $0 < X_{k:n} < T_1 \quad (< T_2 < X_{r:n}), \quad T^* = T_1$,

where the data in parentheses are unobservable.

Let D_j denote the number of failures until $T_j, j = 1,2,$ obviously $D_1 \le D_2$. Then, the likelihood function of this combined HCS is given as follows:

$$L(\vartheta \mid x) = \begin{cases} \dfrac{n!}{(n-k)!}\left[1-F(x_k)\right]^{n-k}\prod_{i=1}^{k} f(x_i); & D_1 = 0,\cdots,k-1, D_2 = k, \\ \dfrac{n!}{(n-D_2)!}\left[1-F(T_2)\right]^{n-D_2}\prod_{i=1}^{D_2} f(x_i); & D_1 = 0,\cdots,k-1, D_2 = 0,\cdots,k-1, \\ \dfrac{n!}{(n-r)!}\left[1-F(x_r)\right]^{n-r}\prod_{i=1}^{r} f(x_i); & D_1 = D_2 = r, \\ \dfrac{n!}{(n-D_1)!}\left[1-F(T_1)\right]^{n-D_1}\prod_{i=1}^{D_1} f(x_i); & D_1 = D_2 = k,\cdots,r-1, \end{cases} \qquad (1)$$

where the parameter $\vartheta \in \Theta$ may be a real vector.

In this paper, we obtain estimation based on combined HCS for the three unknown parameter of the modified Weibull (MW) distribution, as well as the survival time parameters. Section 2 contains some preliminaries. In Section 3, the Bayes estimators of the parameters, the reliability and hazard functions are derived. A real life data set and simulation data results, and data analysis are provided in Sections 4 and 5, respectively.

II. PRELIMINARIES

Suppose the lifetime random variable X has the MW distribution, suggested by Sarhan and Zaindin (2009), with the following probability density function (pdf)

$$f(x) = \left(\alpha + \theta\beta x^{\beta-1}\right)\exp\left(-\alpha x - \theta x^{\beta}\right), \quad x > 0, \qquad (2)$$

and cumulative distribution function (cdf) is

$$F(x) = 1 - \exp\left(-\alpha x - \theta x^{\beta}\right), \quad x > 0, \qquad (3)$$

where $\alpha \ge 0$ is a scale parameter, while $\theta \ge 0$ and $\beta > 0$ are shape parameters such that $\alpha + \theta > 0$.

It is easy to write at mission time t for MW distribution, the reliability and hazard functions $R(t)$ and $H(t)$, respectively, as

$$R(t) = \exp\left(-\alpha t - \theta t^{\beta}\right), \quad t > 0, \qquad (4)$$

and

$$H(t) = \alpha + \theta\beta t^{\beta-1}, \quad t > 0. \qquad (5)$$

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It is very well known that MW distribution generalized exponential distribution (when $\beta = 0$), Rayleigh distribution (when $\alpha = 0$ and $\beta = 2$), linear exponential distribution (when $\beta = 2$ and $\theta = \lambda/2$, $\lambda > 0$) and Weibull distribution (when $\alpha = 0$).

Note that Equation (1) can be combined and they can be written as the following two cases. The likelihood function in Case I is given by

where $x_{D_j} = (x_1, x_2, ..., x_{D_j})$;

$0 < ... < x_{D_1} \le T_1 < x_{D_1+1}$,

$T^* = T_1, D_1 = D_2 = k, ..., r-1$ for $j = 1$,

$0 < ... < x_{D_2} \le T_2 < x_{D_2+1}$,

$T^* = T_2, D_1 = 0,..,k-1; D_2 = 0,..,k-1; D_1 \le D_2$ for $j = 2$.

The following relation will be used throughout this paper: If $n \in N$ and N is the set of natural numbers, then

$$\prod_{i=1}^{n}\left(\alpha + \theta Z_i(\beta)\right) = \sum_{j=0}^{n} \alpha^{n-j}\theta^j \xi_j(n, \beta),$$

where for $j = 0$, $\xi_0(n, \beta) = 1$ and for $1 \le j \le n$,

$$\xi_j(n, \beta) = \sum_{b_1=1}^{n-j+1} Z_{b_1}(\beta) \sum_{b_2=b_1+1}^{n-j+2} Z_{b_2}(\beta) \times \ldots \times \sum_{b_j=b_{j-1}+1}^{n} Z_{b_j}(\beta)$$

$$= \sum_{b_1=1}^{n-j+1} \sum_{b_2=b_1+1}^{n-j+2} \times \ldots \times \sum_{b_j=b_{j-1}+1}^{n} \prod_{i=1}^{j} Z_{b_i}(\beta).$$

III. BAYESIAN ESTIMATION

We obtain Bayes estimators of α , θ , β , $R(t)$ and $H(t)$ based on the observation given in Section 2. By using Equations (2), (3) and expression (8) in Equations (6) and (7), we can obtain the likelihood function in Case I as follows

$$L_1(\alpha, \theta, \beta \mid x_\ell) = \frac{n!}{(n-\ell)!} \sum_{j=0}^{\ell} \alpha^{\ell-j}\theta^j \xi_j(\ell, \beta) \exp\{-\alpha\psi_1(1) - \theta\psi_1(\beta)\}, \tag{10}$$

where $\xi_j(\ell, \beta)$ is given by (9), $Z_i(\beta) = \beta x_i^{\beta-1}$ and

$$\psi_1(\beta) = (n-\ell)x_\ell^\beta + \sum_{i=1}^{\ell} x_i^\beta,$$

and for Case II, it is given by

$$L_2(\alpha, \theta, \beta \mid x_{D_j}) = \frac{n!}{(n-D_j)!} \sum_{j=1}^{D_j} \alpha^{D_j-j}\theta^j \xi_j(D_j, \beta) \exp\{-\alpha\psi_2(1) - \theta\psi_2(\beta)\},$$

where
$$\psi_2(\beta) = (n-D_j)T_j^\beta + \sum_{i=1}^{D_j} x_i^\beta. \tag{13}$$

Estimation of the parameters when β is known

Under the assumption that the parameters α and θ are unknown and independent, we can use the joint prior density function, suggested by Al-Hussaini et al. (2000), that is given by

$$\pi(\alpha, \theta) = \nu_1\nu_2 \exp[-(\nu_1\alpha + \nu_2\theta)], \quad \alpha > 0, \theta > 0,$$

where ν_1 and ν_2 are positive constants.

Then, from (10), (12) and (14), the joint posterior density function of α and θ in Case I is given by

$$\pi_1^*(\alpha, \theta \mid x_\ell) = \sum_{j=1}^{\ell} \frac{\xi_j(\ell, \beta)\alpha^{\ell-j}\theta^j}{J_1(\nu_1, \nu_2, 1, 1)} \exp\{-\alpha(\psi_1(1) + \nu_1) - \theta(\psi_1(\beta) + \nu_2)\}, \tag{15}$$

and for Case II, it is given by

$$\pi_2^*(\alpha, \theta \mid x_{D_j}) = \sum_{j=1}^{D_j} \frac{\xi_j(D_j, \beta)\alpha^{D_j-j}\theta^j}{J_2(\nu_1, \nu_2, 1, 1)} \exp\{-\alpha(\psi_2(1) + \nu_1) - \theta(\psi_2(\beta) + \nu_2)\}, \tag{16}$$

where J_1 and J_2 are the normalizing constants satisfying

$$J_1(\nu_1, \nu_2, \tau_1, \tau_2) = \sum_{j=1}^{\ell} \xi_j(\ell, \beta) \frac{\Gamma(\ell - j + \tau_1)}{(\psi_1(1) + \nu_1)^{\ell-j+\tau_1}} \times \frac{\Gamma(j + \tau_2)}{(\psi_1(\beta) + \nu_2)^{j+\tau_2}}, \tag{17}$$

and

$$J_2(\nu_1, \nu_2, \tau_1, \tau_2) = \sum_{j=1}^{D_j} \xi_j(D_j, \beta) \frac{\Gamma(D_j - j + \tau_1)}{(\psi_2(1) + \nu_1)^{D_j-j+\tau_1}} \times \frac{\Gamma(j + \tau_2)}{(\psi_2(\beta) + \nu_2)^{j+\tau_2}}. \tag{18}$$

(8)

Therefore, the estimations of the parameters α and θ in Cases I are given, respectively, by

$$\tilde{\alpha} = E(\alpha \mid x_\ell) = \frac{J_1(\nu_1, \nu_2, 2, 1)}{J_1(\nu_1, \nu_2, 1, 1)} \quad \text{and} \quad \tilde{\theta} = \frac{J_1(\nu_1, \nu_2, 1, 2)}{J_1(\nu_1, \nu_2, 1, 1)}. \tag{19}$$

From (4), (5), (15) and (16), the Bayes estimators for the reliability function $R(t)$ and the hazard function $H(t)$ (9) are given, respectively, by

$$\tilde{R}(t) = \int_0^\infty \int_0^\infty \exp\left(-\alpha t - \theta t^\beta\right) \pi_1^*(\alpha, \theta \mid x_\ell) d\alpha d\theta = \frac{J_1(\nu_1 + t, \nu_2 + t^\beta, 1, 1)}{J_1(\nu_1, \nu_2, 1, 1)}, \tag{20}$$

and

$$\tilde{H}(t) = \frac{J_1(\nu_1, \nu_2, 2, 1)}{J_1(\nu_1, \nu_2, 1, 1)} + \frac{J_1(\nu_1, \nu_2, 1, 2)}{J_1(\nu_1, \nu_2, 1, 1)} \beta t^{\beta-1}.$$

Putting $\beta = 1$ and $\xi_j(n, 1) = \binom{n}{j}$, $j = 0, 1, ..., n$ in this subsection, yields Bayesian estimators from exponential distribution with parameter $\alpha + \theta$. Setting $\beta = 0$, $\xi_0(n, 0) = 1$ and $\xi_j(n, 0) = 0$, $j = 1, ..., n$ Bayesian estimators from shifted exponential distribution can be obtained. Moreover, putting $\beta = 2$ (11) and $\theta = \lambda/2$, $\lambda > 0$, we obtain Bayesian estimators from linear exponential distribution with parameters α and λ (12), where

$$\xi_j(n, 2) = 2^j \sum_{b_1=1}^{n-j+1} x_{b_1} \sum_{b_2=b_1+1}^{n-j+2} x_{b_2} \times \ldots \times \sum_{b_j=b_{j-1}+1}^{n} x_{b_j}.$$

Similarly, the Bayes estimators for α , θ , $R(t)$ and $H(t)$ in Cases II are obtained, if J_1 is replaced by J_2 in Equations (19), (20) and (21), respectively.

Estimation of the parameters α , θ and β

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In this case, we note that a conjugate family of continuous joint prior does not exist, so we use the Soland's method. Soland (1969) considered a family of joint prior distributions that places continuous distributions on the scale parameter and discrete distributions on the shape parameter. We assume that the parameter β has a discrete distribution $\Pr(\beta=\beta_i)=\eta_i$, $\sum_i^{\ell}\eta_i=1$ for $i=1,\ldots,k$ and suppose that α and θ given $\beta=\beta_i$ have natural conjugate prior gamma distribution with density function given by

$$\pi(\alpha,\theta\,|\,\beta=\beta_i)=\nu_i\nu_i^*\exp\left[-(\nu_i\alpha+\nu_i^*\theta)\right],\quad \alpha>0,\theta>0, \tag{22}$$

where ν_i and ν_i^* are chosen so as to reflect prior beliefs on α and θ given $\beta=\beta_i$. Then, from (10) and (22), the conditional posterior density function of α and θ in Case I is given by

$$\pi_1^*(\alpha,\theta\,|\,\beta=\beta_i,x_\ell)=\sum_{j=0}^{\ell}\frac{\xi_j(\ell,\beta_i)\alpha^{\ell-j}\theta^j}{A_1(\nu_i,\nu_i^*,1,1)}\exp\{-\alpha(\nu_i+\psi_1(1))-\theta(\nu_i^*+\psi_1(\beta_i))\}, \tag{23}$$

and for Case II, from (12) and (22), it is given by

$$\pi_2^*(\alpha,\theta\,|\,\beta=\beta_i,x_{D_j})=\sum_{j=0}^{D_j}\frac{\xi_j(D_j,\beta_i)\alpha^{D_j-j}\theta^j}{A_2(\nu_i,\nu_i^*,1,1)}\exp\{-\alpha(\nu_i+\psi_2(1))-\theta(\nu_i^*+\psi_2(\beta_i))\}, \tag{24}$$

where

$$A_1(\nu_i,\nu_i^*,\tau_1,\tau_2)=\sum_{j=0}^{\ell}\xi_j(\ell,\beta_i)\frac{\Gamma(\ell-j+\tau_1)}{(\nu_i+\psi_1(1))^{\ell-j+\tau_1}}\frac{\Gamma(j+\tau_2)}{(\nu_i^*+\psi_1(\beta_i))^{j+\tau_2}}.$$

and

$$A_2(\nu_i,\nu_i^*,\tau_1,\tau_2)=\sum_{j=0}^{D_j}\xi_j(D_j,\beta_i)\frac{\Gamma(D_j-j+\tau_1)}{(\nu_i+\psi_2(1))^{\ell-j+\tau_1}}\frac{\Gamma(j+\tau_2)}{(\nu_i^*+\psi_2(\beta_i))^{j+\tau_2}}.$$

On applying the discrete version of Bayes' theorem, the marginal posterior probability distribution of β_j is given by

$$P_i=\begin{cases}P_{1i}=Pr(\beta=\beta_i\,|\,x_\ell)=B_1^{-1}\eta_i\nu_i\nu_i^*A_1(\nu_i,\nu_i^*,1,1), & \text{for Case I},\\ P_{2i}=Pr(\beta=\beta_i\,|\,x_{D_j})=B_2^{-1}\eta_i\nu_i\nu_i^*A_2(\nu_i,\nu_i^*,1,1), & \text{for Case II},\end{cases} \tag{25}$$

where

$$B_1=\sum_{i=1}^{k}\eta_i\nu_i\nu_i^*A_1(\nu_i,\nu_i^*,1,1)\quad\text{and}\quad B_2=\sum_{i=1}^{k}\eta_i\nu_i\nu_i^*A_2(\nu_i,\nu_i^*,1,1). \tag{26}$$

The Bayes estimators for the parameters α, θ and β, using the posterior pdfs (23),(24) and (25), in Cases I, is given by

$$\tilde{\alpha}=\int_0^\infty\!\!\int_0^\infty\sum_{i=1}^{k}P_{1i}\alpha\pi_1^*(\alpha,\theta\,|\,\beta=\beta_i,x_\ell)d\alpha d\theta=\sum_{i=1}^{k}P_{1i}\frac{A_1(\nu_i,\nu_i^*,2,1)}{A_1(\nu_i,\nu_i^*,1,1)},$$

$$\tilde{\theta}=\sum_{i=1}^{k}P_{1i}\frac{A_1(\nu_i,\nu_i^*,1,2)}{A_1(\nu_i,\nu_i^*,1,1)}\quad\text{and}\quad\tilde{\beta}=\sum_{i=1}^{k}P_{1i}\beta_i. \tag{27}$$

Similarly, the Bayes estimators $\tilde{R}(t)$ and $\tilde{H}(t)$ of the reliability and hazard functions $R(t)$ and $H(t)$ are given, respectively, by

$$\tilde{R}(t)=\int_0^\infty\!\!\int_0^\infty\sum_{i=1}^{k}P_{1i}\exp\left(-\alpha t-\theta t^{\beta_i}\right)\pi_1^*(\alpha,\theta\,|\,\beta=\beta_i,x_\ell)d\alpha d\theta$$

$$=\sum_{i=1}^{k}P_{1i}\frac{A_1\left(\nu_i+t,\nu_i^*+t^{\beta_i},1,1\right)}{A_1(\nu_i,\nu_i^*,1,1)},$$

and

$$\tilde{H}(t)=\sum_{i=1}^{k}P_{1i}\frac{A_1(\nu_i,\nu_i^*,2,1)}{A_1(\nu_i,\nu_i^*,1,1)}+\sum_{i=1}^{k}P_{1i}\frac{A_1(\nu_i,\nu_i^*,1,2)}{A_1(\nu_i,\nu_i^*,1,1)}\beta_i t^{\beta_i-1}. \tag{29}$$

An alternative method for obtaining the values ν_i and ν_i^* can be based on the expected value of the reliability function R(t) conditional on $\beta=\beta_i$, which is given by

$$E_{\alpha,\theta|\beta_i}[R(t)\,|\,\beta=\beta_i]=\nu_i\nu_i^*\int_0^\infty\!\!\int_0^\infty\exp\left[-\alpha(\nu_i+t)-\theta(\nu_i^*+t^{\beta_i})\right]d\alpha d\theta$$

$$=\frac{\nu_i\nu_i^*}{(\nu_i+t)(\nu_i^*+t^{\beta_i})}. \tag{30}$$

Thus, the values of ν_i and ν_i^* can be obtained numerically for each value β_i and for two prior probabilities values $R(t=t_1)$ and $R(t=t_2)$. Martz and Waller (1982, p. 105) have discussed, an alternative method when there are no prior beliefs. A nonparametric procedure can be used to estimate the corresponding two different values of $R(t)$.

IV. NUMERICAL RESULTS

Now, we consider a real life data set which given in Nelson (1982) to illustrate the methods proposed in the previous sections. These data which was also used in Lawless (1982, p. 185), concerning the data on time to breakdown of an insulating fluid between electrodes at a voltage of 34 kV (minutes). The 19 times to breakdown are contained in the sample (*)

0.190	0.780	0.960	1.310	2.780
3.160	4.150	4.670	4.850	6.500
7.350	8.010	8.270	12.060	31.750
32.520	33.910	36.710	72.892	

When scale and shape parameters are unknown, using gamma prior for α and θ and a discrete prior (Soland's method) for the shape parameter β. The hyperparameters ν_i and ν_i^* of the gamma prior $\Gamma(1,.)$ and the values of β_i can be obtained by the following steps:

1. Based on the real data set (*) of size $n=19$, using a nonparametric procedure $R(t_j=X_{j:n})=(n-j+0.625)/(n+0.25)$, $j=1,2,\ldots,n$ to estimate two

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values of the reliability function for any two different times t_1 and t_2, see Martz and Waller (1982, p. 105). If we choose $(t_1, t_2) = (0.780, 4.850)$, then $(R(t_1), R(t_2)) = (0.916, 0.604)$.

2. Let β_i takes the values $0.1(0.1)1$, $0.6(0.1)1.5$ and $1.1(0.1)2$, using step 1 and Eq. (30), then the values of ν_i and ν_i^* for each value β_i, $i = 1,\ldots,10$, can be obtained numerically (Newton-Raphson method). The values of the hyperparameters and the posterior mass function derived for each β_i are presented in Table 1. By using the results in Subsection 3.2, the Bayes estimates of α, θ, β, $R(t)$, and $H(t)$ are computed and the results are displayed in Tables 2 and 3 in Cases I and II, respectively.

Table 1: Values of hyperparameter (ν_i, ν_i^*), δ_i and the posterior probabilities P_i for each β_i.

i	δ_i	β_i	ν_i	ν_i^*	P_i
1	0.1	0.1	6.258	46.202	0.015
2	0.1	0.2	6.340	44.355	0.032
3	0.1	0.3	6.447	42.445	0.051
4	0.1	0.4	6.588	40.458	0.072
5	0.1	0.5	6.776	38.384	0.096
6	0.1	1.6	7.034	36.212	0.122
7	0.1	0.7	7.391	33.933	0.149
8	0.1	0.8	7.902	31.546	0.168
9	0.1	0.9	8.659	29.056	0.165
10	0.1	1.0	9.838	26.487	0.131
1	0.1	0.6	13.11	24.667	0.430
2	0.1	0.7	10.64	42.110	0.221
3	0.1	0.8	9.846	60.189	0.127
4	0.1	0.9	9.433	80.415	0.079
5	0.1	1.0	9.179	103.474	0.051
6	0.1	1.1	9.009	129.976	0.034
7	0.1	1.2	8.888	160.563	0.023
8	0.1	1.3	8.799	195.959	0.016
9	0.1	1.4	8.732	236.992	0.011
10	0.1	1.5	8.679	284.622	0.008
1	0.1	1.1	9.101	129.976	0.211
2	0.1	1.2	8.487	160.563	0.168
3	0.1	1.3	8.479	195.959	0.136
4	0.1	1.4	8.475	236.992	0.112
5	0.1	1.5	8.473	284.622	0.092
6	0.1	1.6	8.471	339.961	0.077
7	0.1	1.7	8.469	404.309	0.065
8	0.1	1.8	8.467	479.173	0.054
9	0.1	1.9	8.466	566.317	0.045
10	0.1	2.0	8.465	667.793	0.038

Table 2: Estimates of α, θ, $R(t)$ and $H(t)$ when the experiment terminate at $T^* = X_{k:19}$ or $T^* = X_{r:19}$.

k	β_i	$\tilde{\alpha}$	$\tilde{\theta}$	$\tilde{\beta}$	$\tilde{R}(t)$	$\tilde{H}(t)$
8	0.1(0.1)1	0.084	0.042	0.718	0.113	0.872
10		0.085	0.041	0.719	0.114	0.872
12		0.091	0.041	0.723	0.119	0.867
8	0.6(0.1)1.5	0.070	0.036	0.920	0.115	0.891
10		0.078	0.027	0.927	0.110	0.885
12		0.077	0.026	0.935	0.114	0.882
8	1.1(0.1)2	0.074	0.029	1.320	0.113	0.893
10		0.080	0.030	1.332	0.122	0.881
12		0.074	0.027	1.327	0.111	0.895
r						
14	0.1(0.1)1	0.086	0.040	0.719	0.114	0.872
16		0.051	0.042	0.684	0.079	0.904
18		0.057	0.040	0.691	0.084	0.900
14	0.6(0.1)1.5	0.082	0.033	0.876	0.088	0.890
16		0.069	0.019	0.904	0.080	0.842
18		0.071	0.022	0.927	0.092	0.927
14	1.1(0.1)2	0.073	0.024	1.322	0.106	0.898
16		0.050	0.015	1.234	0.069	0.931
18		0.053	0.016	1.249	0.073	0.929

- *Example 2 (simulated data)*

In this example, a simulation study was conducted to illustrate our results in Section 3 and to compare the performance of the presented Bayes estimators in different situations.

(The case of known β)

By choosing the values of the hyperparameters $(\nu_1, \nu_2) = (0.5, 0.25)$ we generate the values $(\alpha, \theta) = (0.499, 0.249)$ from $\Gamma(1, \nu_i)$, $i = 1, 2$. Using these generated values with $\beta = 1.5, 2$ and 3, we generate random samples of size n from the MW pdf in (2). The Bayes estimates of α, β, $R(t)$, and $H(t)$ are computed through $(19)-(21)$. This process are repeated 1000 times, and the mean square error (MSE) was calculated. The results are presented in Tables 4 and 5.

Table 3: Estimates of α, θ, $R(t)$ and $H(t)$ when the experiment terminate at $T^* = T_1$ or $T^* = T_2$.

T_1	β_i	$\tilde{\alpha}$	$\tilde{\theta}$	$\tilde{\beta}$	$\tilde{R}(t)$	$\tilde{H}(t)$
8	0.1(0.1)1	0.102	0.057	0.834	0.149	0.841
8.2		0.113	0.057	0.834	0.159	0.832
10		0.106	0.056	0.835	0.153	0.838
8	0.6(0.1)1.5	0.071	0.053	1.078	0.129	0.872

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8.2		0.076	0.054	1.095	0.136	0.867
10		0.074	0.053	1.082	0.132	0.870
8	1.1(0.1)2	0.070	0.035	1.388	0.119	0.890
8.2		0.074	0.037	1.355	0.125	0.885
10		0.073	0.035	1.333	0.121	0.888
T_2						
20	0.1(0.1)1	0.063	0.058	0.820	0.110	0.877
32		0.041	0.063	0.790	0.090	0.894
40		0.051	0.058	0.814	0.079	0.889
20	0.6(0.1)1.5	0.053	0.050	0.975	0.102	0.894
32		0.037	0.061	0.870	0.089	0.901
40		0.045	0.050	0.939	0.092	0.902
20	1.1(0.1)2	0.059	0.024	1.238	0.089	0.913
32		0.049	0.017	1.207	0.069	0.930
40		0.053	0.018	1.224	0.076	0.925

Table 4: MSEs of the estimates of α, θ, $R(t)$ and $H(t)$ of case I.

β	k	$\tilde{\alpha}$	$\tilde{\theta}$	$\tilde{R}(t)$	$\tilde{H}(t)$
1.5	8	0.030	0.133	0.080	0.231
	10	0.024	0.100	0.078	0.158
	12	0.019	0.079	0.077	0.113
2	8	0.040	0.506	0.081	2.527
	10	0.038	0.291	0.080	1.445
	12	0.037	0.154	0.080	0.750
3	8	0.049	0.751	0.089	3.897
	10	0.040	0.423	0.089	2.165
	12	0.038	0.161	0.088	1.934
	r				
1.5	14	0.018	0.068	0.076	0.087
	16	0.017	0.057	0.076	0.066
	18	0.017	0.053	0.076	0.062
2	14	0.037	0.089	0.080	0.425
	16	0.036	0.060	0.080	0.356
	18	0.036	0.055	0.079	0.255
3	14	0.037	0.106	0.088	1.436
	16	0.031	0.093	0.087	1.252
	18	0.016	0.078	0.061	0.790

(The case of unknown α, θ and β)

Based on Soland's method and the values of the parameter β which used in the generation of previous samples, the values of $\beta = 1.5, 2$ and 3 are chosen as $0.5(1)1.4$, $1.5(2)2.4$ and $2.5(3)3.4$, respectively, each with probability 0.1. By using the same procedure in Example 1, we obtained the values of the hyperparameters ν_i and ν_i^* for a given values of $\beta_i, i = 1,2,\cdots,10$. Based on the above combined HCS schemes, using the results presented earlier in Subsection 3.2, the Bayes estimates of α, θ, β, $R(t)$ and $H(t)$ are computed. The MSEs of Bayes estimates are presented in Tables 6 and 7.

Table 5: MSEs of the estimates of α, θ, $R(t)$ and $H(t)$ of case II.

β	T_1	$\tilde{\alpha}$	$\tilde{\theta}$	$\tilde{R}(t)$	$\tilde{H}(t)$
1.5	0.5	0.030	0.179	0.081	0.315
	0.6	0.024	0.146	0.079	0.244
	0.7	0.020	0.120	0.078	0.187
2	0.5	0.032	0.185	0.081	0.328
	0.6	0.024	0.148	0.080	0.245
	0.7	0.020	0.122	0.079	0.190
3	0.5	0.091	5.151	0.109	7.926
	0.6	0.070	2.532	0.096	4.566
	0.7	0.052	1.249	0.086	2.652
	T_2				
1.5	0.8	0.024	0.111	0.078	0.168
	0.9	0.017	0.097	0.077	0.138
	1.0	0.012	0.096	0.077	0.137
2	0.8	0.028	0.115	0.078	0.175
	0.9	0.018	0.096	0.077	0.137
	1.0	0.017	0.091	0.077	0.129
3	0.8	0.045	0.589	0.081	1.436
	0.9	0.043	0.394	0.080	1.252
	1.0	0.042	0.200	0.080	1.190

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Table 6: MSEs of the estimates of α, θ, β, $R(t)$ and $H(t)$ of case I.

T_1	β_i	$\tilde{\alpha}$	$\tilde{\theta}$	$\tilde{\beta}$	$\tilde{R}(t)$	$\tilde{H}(t)$
0.5	0.1(0.1)1	0.031	0.083	0.004	0.069	0.228
0.6		0.028	0.078	0.004	0.061	0.199
0.7		0.026	0.071	0.003	0.053	0.142
0.5	1.5(0.1)2.4	0.044	0.098	0.004	0.072	0.237
0.6		0.029	0.088	0.004	0.066	0.211
0.7		0.027	0.070	0.003	0.061	0.184
0.5	2.5(0.1)3.4	0.051	0.099	0.007	0.090	0.421
0.6		0.043	0.099	0.005	0.083	0.361
0.7		0.038	0.098	0.004	0.081	0.331
T_2						
0.8	0.1(0.1)1	0.025	0.062	0.002	0.050	0.135
0.9		0.022	0.056	0.001	0.048	0.099
1.0		0.018	0.051	0.000	0.048	0.087
0.9	1.5(0.1)2.4	0.033	0.078	0.003	0.057	0.177
1.0		0.026	0.075	0.002	0.051	0.155
1.0		0.025	0.075	0.002	0.051	0.155
0.8	2.5(0.1)3.4	0.036	0.089	0.004	0.080	0.330
0.9		0.033	0.089	0.004	0.066	0.311
1.0		0.028	0.088	0.003	0.055	0.305

V. CONCLUSION

In this paper, we have considered the Bayesian approaches to estimate the three unknown parameters as well as the reliability and hazard functions for MW distribution, based on combined HCS. From the results, we observe the following:

1. It may be noted from the results in all the tables that the MSEs for Bayes method decrease with decrease β.

2. Tables 4-7 show that the MSEs for Bayes method decreases with increasing k, r, T_1 and T_2.

3. One of the useful results of our work is the use Equation (8) which gives a closed form expression for results in estimating.

4. It is seen for the case of the unknown shape and scale parameters that a discrete distribution for the shape parameter β give a closed form expression for the posterior pdf. Also, the equal probabilities chosen in the discrete distributions caused an element of uncertainty, which can be desirable in some cases.

Table 7: MSEs of the estimate of α, θ, β, $R(t)$ and of $H(t)$ case II.

k	β_i	$\tilde{\alpha}$	$\tilde{\theta}$	$\tilde{\beta}$	$\tilde{R}(t)$	$\tilde{H}(t)$
8	0.1(0.1)1	0.030	0.115	0.002	0.080	0.089
10		0.030	0.076	0.002	0.079	0.081
12		0.026	0.068	0.001	0.061	0.079
8	1.5(0.1)2.4	0.046	0.191	0.006	0.089	0.119
10		0.031	0.185	0.005	0.080	0.088
12		0.030	0.183	0.004	0.074	0.085
8	2.5(0.1)34	0.051	0.389	0.007	0.088	0.282
10		0.043	0.306	0.006	0.081	0.238
12		0.033	0.288	0.005	0.076	0.221
r						
14	0.1(0.1)1	0.018	0.047	0.001	0.049	0.075
16		0.009	0.029	0.000	0.044	0.052
18		0.005	0.023	0.000	0.031	0.039
14	1.5(0.1)2.4	0.027	0.081	0.003	0.066	0.081
16		0.019	0.070	0.002	0.065	0.055
18		0.013	0.068	0.001	0.057	0.047
14	2.5(0.1)34	0.033	0.229	0.004	0.070	0.196
16		0.022	0.217	0.003	0.069	0.164
18		0.017	0.178	0.003	0.061	0.161

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