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Recursive Approximation Method for Solving Constrained Nonlinear Optimal Control Problems Using Legendre Polynomials

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***Abstract:** A computational method is presented to solve a nonlinear quadratic optimal control problems subject to terminal state constraints, path inequality constraints on both the state and the control variables. The method is based on using a recursive approximation technique to replace the original constrained nonlinear dynamic system by a sequence of constrained linear time-varying systems. Then each of constrained linear time-varying quadratic optimal control problems is approximated by a quadratic programming problem by parameterizing each of the state variable by a finite length Legendre polynomials with unknown parameters. To show the effectiveness of the proposed method, simulation results of two constrained nonlinear optimal control problems are presented.*

***Index Terms:** Nonlinear constrained quadratic optimal control problem; Iterative Technique; Legendre polynomials; State parameterization.*

I. INTRODUCTION

One of the well known methods to solve the constrained nonlinear optimal control problems is to convert it into a mathematical programming one. This approach of handling the optimal control problem is known as direct method. The direct method can be employed by either discretizating or parameterizing of the control and the state variables. Direct methods have been proposed to handle the constrained nonlinear optimal control problems in many research papers, for example: Vlassenbroeck [1] used the control-state parameterization using Chebyshev polynomials to handle the constrained nonlinear optimal control problem by converting it into nonlinear mathematical programming problem. Frick and Stech [2] used the Walsh functions to solve the nonlinear optimal control problems subject to saturation constraints on the control. Goh and Teo [3], Troch *et al.* [4] proposed a method to handle the constrained nonlinear optimal control problems using control parameterization. Jaddu [5-7] proposed a method that is based on the second method of quasilinearization and the state parameterization via Chebyshev polynomials to handle the nonlinear optimal control problems subject to state and control saturation constraints. Recently, several researchers proposed methods to solve the nonlinear optimal control problem based on the Approximate sequence Riccati equation ASRE [8-11]. In these methods the nonlinear dynamic state equation is replaced by a sequence of linear time-varying state equations.

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In [12] a method was presented to solve the unconstrained nonlinear optimal control problem by solving sequence of quadratic programming problems instead of solving sequence of Riccati equations. In this work, we will extend the method proposed in [12] and the work in [8-11] to handle constrained nonlinear quadratic optimal control problem. The constraints considered in this work are the terminal state constraints and inequality constraints on the state and control variables. The method is based on converting the constrained nonlinear optimal control problem into sequence of quadratic programming problems using Legendre polynomials.

II. PROBLEM STATEMENT

The optimal control problem treated in this paper can be stated as follows: Find an optimal controller $u^*(t)$ that minimizes the following cost function

$$J = x(t_f)^T S x(t_f) + \int_0^{t_f} (x^T Q x + u^T R u) dt \qquad (1)$$

subject to the following constraints:

- Dynamic system state equations and initial conditions

$$\dot{x} = f(x(t), u(t), t) \qquad x(0) = x_0 \qquad (2)$$

- Linear terminal state constraints

$$F\, x(t_f) = X_f \qquad (3)$$

- Path inequality constraints on the state and control

$$x(t) \le X_{max}\ , x(t) \ge X_{min}\ , u(t) \le U_{max}\ , u(t) \ge U_{min} \qquad (4)$$

where Q, S are positive semidefinite matrices, R is a positive definite matrix, $x \in R^n$ is the state vector, $u \in R^m$ is the control vector, $x_0 \in R^n$ is the initial condition vector, F is an $s \times n$ constant matrix, $X_f \in R^s$ is the terminal state, f is a nonlinear continuous differentiable function with respect to all its arguments $(x(t), u(t), t)$. We will assume that: $m \le n$, $X_{max}, X_{min}, U_{min}$ and U_{max} are constant quantities and t_f is fixed.

This problem (1)-(4) ,will be solved by converting it into a sequence of constrained linear time-varying quadratic optimal problems. The solution is based on using the iteration technique; which will replace the nonlinear dynamic state equations into equivalent linear time-varying state equations.

The proposed algorithm to solve the problem consists of the following steps:

1. The first step, is to convert the constrained nonlinear quadratic optimal control problem into a sequence of constrained linear time-varying quadratic optimal control problems by using the method of [8-12].
2. The second step is to convert each of these problems into quadratic programming problem by using the Legendre polynomials to parameterize the state variables.
3. The third step is to solve the quadratic programming problem . Then the solution of the state and the control variables are used in the first step for new iteration.

III. RECURSIVE APPROXIMATION

The optimal control problem described in section II can be replaced by the following sequence of problems by applying the approximation described in [8-11]:
For k ≥ 1, minimize

$$J^{[k]} = x(t_f)^{[k]^T} S x(t_f)^{[k]} + \int_0^{t_f} \left(x^{[k]^T} Q x^{[k]} + u^{[k]^T} R u^{[k]} \right) dt \quad (5)$$

subject to the following linear time-varying state equations,

$$\dot{x}^{[k]} = A\left(x^{[k-1]}\right)x^{[k]} + B\left(x^{[k-1]}\right)u^{[k]}, \; x^{[k]}(0) = x_0 \quad (6)$$

and to the terminal constraints

$$F \; x(t_f)^{[k]} = X_f \quad (7)$$

and to the state and control variables inequality constraints

$$x(t)^{[k]} \le X_{max} \,, x(t)^{[k]} \ge X_{min} \,, u(t)^{[k]} \le U_{max} \,, \; u(t)^{[k]} \ge U_{min} \quad (8)$$

The constrained linear time-varying quadratic optimal control problems (5)-(8) are solved recursively by converting each problem into a quadratic programming problem by applying the state parameterization using Legendre polynomials. And since Legendre polynomials are orthogonal on the interval $\tau \in [-1,1]$, it is necessary to transform the time interval of the original problem $t \in [0, t_f]$ into $\tau \in [-1,1]$. This will transform (5)-(8) into the problem: for $k \ge 1$,
Minimize

$$J^{[k]} = x(1)^{[k]^T} S x(1)^{[k]} + \frac{t_f}{2}\int_{-1}^{1} (x^{[k]^T} Q x^{[k]} + u^{[k]^T} R u^{[k]}) d\tau \quad (9)$$

subject to the following constraints

$$\frac{dx^{[k]}}{d\tau} = \frac{t_f}{2}\left[A\left(x^{[k-1]}(\tau)\right)x^{[k]} + B\left(x^{[k-1]}(\tau)\right)u^{[k]}\right]$$
$$x^{[k]}(-1) = x_0 \quad (10)$$

$$F \; x(1)^{[k]} = X_f \quad (11)$$

$$x(\tau)^{[k]} \le X_{max} \quad x(\tau)^{[k]} \ge X_{min} \quad ,u(\tau)^{[k]} \le U_{max} \quad u(\tau)^{[k]} \ge U_{min} \quad (12)$$

IV. PROBLEM REFORMULATION

Each of the constrained linear time-varying quadratic optimal control problems (9)-(12) can be solved by converting it into a quadratic programming problems by applying the state parameterization. For a detailed description of the state parameterization method see [5,12].

A) System state equation approximation:

According to the method of [5,12], some state variables are approximated by Legendre series of unknown parameters. Then the remaining state variables and the control variables are expressed in term of the unknown parameters from the state equation (10). The state and control variables approximated by Legendre series of finite length N with unknown parameters can be expressed as:

$$x_j^{[k]} = \sum_{i=0}^{N} a_i^{(j)} P_i(\tau) \quad j = 1,2,..,n \quad (13)$$
$$u_l^{[k]} = \sum_{i=0}^{N} b_i^{(l)} P_i(\tau) \quad l = 1,2,..,m \quad (14)$$

where $k = 0,1,2,\ldots$ is the iteration sequence number, a's and b's are the unknown parameters, b's are function of a's and $P(\tau)$ is a vector of Legendre polynomials of the first kind. (13)-(14) can be rewritten in matrix form as follows:

$$x^{[k]} = \alpha P(\tau) \quad (15)$$
$$u^{[k]} = \beta P(\tau) \quad (16)$$

where:

$$\alpha = \begin{bmatrix} a_0^{(1)} & a_1^{(1)} & \ldots & a_N^{(1)} \\ a_0^{(2)} & a_1^{(2)} & \ldots & a_N^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ a_0^{(n)} & a_1^{(n)} & \ldots & a_N^{(n)} \end{bmatrix} \quad (17)$$

$$\beta = \begin{bmatrix} b_0^{(1)} & b_1^{(1)} & \ldots & b_N^{(1)} \\ b_0^{(2)} & b_1^{(2)} & \ldots & b_N^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ b_0^{(m)} & b_1^{(m)} & \ldots & b_N^{(m)} \end{bmatrix} \quad (18)$$

B) Performance index approximation:

The next step in converting the optimal control problem under consideration into a quadratic programming problem is to approximate the performance index (9). Substituting (15) and (16) in (9) yields

$$\hat{J}^{[k]} = [P^T(1)\alpha^T S \alpha P(1)] + \frac{t_f}{2}\int_{-1}^{1} (P^T(\tau)\alpha^T Q \alpha P(\tau) + P^T(\tau)\beta^T R \beta P(\tau))d\tau \quad (19)$$

where $\hat{J}^{[k]}$ is the approximated performance index of $J^{[k]}$.

The integration part of (19) can be obtained using theorem 1 in [12], and therefore (19) can be reformulated and written as

$$\hat{J}^{[k]} = P^T(1)\alpha^T S \alpha P(1) + \frac{t_f}{2}\sum_{i=1}^{N+1} \frac{2}{2i-1}(m_{ii} + z_{ii}) \quad (20)$$

where m_{ii} and z_{ii} are the diagonal elements of the symmetrical matrices $M = \alpha^T Q \alpha$ and $Z = \beta^T R \beta$ respectively.

The performance index in (19) or (20) can be rewritten as follows

$$\hat{J}^{[k]} = \frac{1}{2} a^T H a \quad (21)$$

where $a^T = [a_0^{(1)} a_1^{(1)} \dots a_N^{(1)}\ a_0^{(2)} a_1^{(2)} \dots a_N^{(2)} \dots a_0^{(z)} \dots a_N^{(z)}]$ is the unknown parameter vector, H is a positive definite Hessian matrix and z is the number the directly approximated states.

C) Time-varying matrices $A(x^{[k-1]}(\tau))$ and $B(x^{[k-1]}(\tau))$ approximation:

Equation (10) shows that the two matrices $A(x^{[i-1]}(\tau))$ and $B(x^{[i-1]}(\tau))$ are a function of τ, therefore it is necessary to express every τ dependant element in both matrices in terms of a Legendre series of known parameters. To this end, let $A_{jl}(\tau) = g(x^{[i-1]}(\tau), \tau)$ be the (j, l) element of the matrix $A(x^{[i-1]}(\tau))$ where $x^{[i-1]}(\tau)$ is the nominal trajectory of the previous iteration. Then the term $A_{jl}(\tau)$ can be expressed in terms of a Legendre series of known parameters of the form

$$A_{jl}(\tau) = \sum_{i=0}^{M} W_i P_i(\tau) \tag{22}$$

where the coefficients W_i are given by [13] as

$$W_i = \frac{2i+1}{2} \int_{-1}^{1} g(\tau) P_i(\tau) d\tau \tag{23}$$

The same approximation can be done for the matrix $B(x^{[k-1]}(\tau))$.

D) Initial and terminal state constraints approximation:

Using the initial value property of Legendre polynomials at $\tau = -1$, the initial condition vector can be approximated as follows

$$a_0^{(j)} - a_1^{(j)} + a_2^{(j)} - a_3^{(j)} + \dots + (-1)^N a_N^{(j)} - x_j(-1) = 0$$
$$j = 1,2,\dots,n \tag{24}$$

where $x_j(-1) = x_0$. The same procedure can be applied to approximate the terminal state vector. By using the final value property of Legendre polynomials at $\tau = 1$, the following approximation of the terminal state vector can be obtained

$$F\ \alpha\, P(1) = X_f \tag{25}$$

where

$$P^T(1) = [1\ 1 \dots 1] \tag{26}$$

E) Path inequality constraints approximation:

Many methods have been proposed to handle inequality constraints on state or control variables. One method is to add a slack variable to the inequality constraint to convert them into equality constraints. This method was used by [6]. However this method has two drawbacks: The first, adding a slack variable would convert the linear problem into a nonlinear one. The second drawback is the increase in the system dimension as a result of the increase in unknown parameters.

Another method used in [1,14-16] is to discretize the time interval $\tau \in [-1,1]$ with $r + 1$ discrete points, and satisfy the constraints at each point. By this, every continuous constraint is replaced by $r + 1$ constraints. To avoid the drawbacks of the slack variables method, we will adopt this method in this work. Mathematically, the time interval $\tau \in [-1,1]$ is discretized as follows

$$-1 = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_r = 1 \tag{27}$$

Therefore, each of the continuous control saturation constraints is replaced by $r + 1$ finite dimension inequality constraints. Using Legendre polynomials, the $r + 1$ constraints are given by

$$\sum_{i=0}^{N} b_i^{(l)} P_i(\tau_s) \le U_{max} \tag{28}$$
$$-\sum_{i=0}^{N} b_i^{(l)} P_i(\tau_s) \le -U_{min} \tag{29}$$

whereas the state saturation constraints are given by

$$\sum_{i=0}^{N} a_i^{(j)} P_i(\tau_s) \le X_{max} \tag{30}$$
$$-\sum_{i=0}^{N} a_i^{(j)} P_i(\tau_s) \le -X_{min} \tag{31}$$

The difficult constrained nonlinear quadratic optimal control problem is converted into a sequence of standard constrained quadratic programming problems that can be restated as follows:

$$min_a\ \ \frac{1}{2} a^T H a \tag{32}$$

subject to

$$F_1 a = b_1 \tag{33}$$
$$F_2 a \le b_2 \tag{34}$$

where the equality constraints are due to initial conditions, terminal state constraints, and in some cases unsatisfied state equations. While the inequality constraints are due to inequality constraints on the control and the state variables.

The standard quadratic programming problem (32)-(34) can be solve using any available software package. In this work, we use the active set method in MATLAB software to solve this problem

To solve the original nonlinear problem (1) -(4), we need to solve constrained linear time-varying quadratic optimal control problems (9)-(12) iteratively

V. COMPUTATIONAL RESULTS

- **Example1: Van der Pol oscillator problem :**

Find an optimal controller $u^*(t)$ that minimizes the following performance index

$$J = \frac{1}{2} \int_0^5 (x_1^2 + x_2^2 + u^2) dt \tag{35}$$

subject to: The dynamic state equations and initial conditions

$$\dot{x}_1 = x_2 \quad x_1(0) = 1 \tag{36}$$
$$\dot{x}_2 = -x_1 + (1 - x_1^2) x_2 + u \quad x_2(0) = 0 \tag{37}$$

terminal state constraints

$$x_1(5) = -1 \tag{38}$$
$$x_2(5) = 0 \tag{39}$$

and inequality control constraints

$$|u(t)| \le 0.75 \tag{40}$$

This problem can be reformulated and restated using the proposed method as follows: Find an optimal control $u^*(t)$ that minimizes the following performance index:

$$J^{[k]} = \frac{5}{4} \int_{-1}^{1} (\left(x_1^{[k]}\right)^2 + \left(x_2^{[k]}\right)^2 + \left(u^{[k]}\right)^2) d\tau \tag{41}$$

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subject to:

$$\frac{dx_1^{[k]}}{d\tau} = \frac{5}{2}x_2^{[k]} \quad , x_1^{[k]}(-1) = 1 \qquad (42)$$

$$\frac{dx_2^{[k]}}{d\tau} = \frac{5}{2}\left[-x_1^{[k]} + \left(1 - \left(x_1^{[k-1]}\right)^2\right)x_2^{[k]} + u^{[k]}\right],$$

$$x_2^{[k]}(-1) = 0 \qquad (43)$$

$$x_1^{[k]}(1) = -1 \qquad (44)$$
$$x_2^{[k]}(1) = 0 \qquad (45)$$
$$\left|u(\tau)^{[k]}\right| \le 0.75 \qquad (46)$$

After changing the time interval $t \in [0,5]$ to the interval $\tau \in [-1,1]$, $x_1(\tau)$ is approximated by a 9th order Legendre series, $x_2(\tau)$ is determined from (42) while $u(\tau)$ is determined from (43). The inequality control constrains are satisfied at 21 equally spaced points in the interval $[-1,1]$, namely at

$\tau = -1, -0.9, -0.8, -0.7, \ldots, 0.7, 0.8, 0.9, 1$

Table 1 illustrates the results of the optimal values of the cost function J for several iterations.

Table 1 approximated optimal value J

Iteration k	J
0	1.854040831
1	2.381162382
2	2.236824541
3	2.292189099
4	2.263384111
5	2.277806397

This problem was solved by Bashein and Enns [17], and they obtained $J = 2.1439199$, after seven iteration. This problem also solved by Jaddu [5] using quasilinearization and state parameterization using Chebyshev polynomials and J was found to be 2.1443893 after seven iteration. Figures 1 shows the approximated optimal control and state trajectories of the Van der Pol oscillator

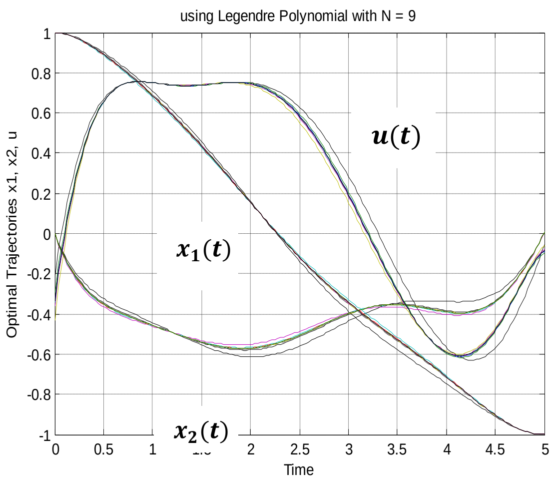

Fig. 1. Optimal control and state trajectories using Legendre Polynomials

- **Example 2: Container crane problem:**

In this example, we consider a practical and complex problem; the container crane problem [18]. Find an optimal controller $u^*(t)$ that minimizes the performance index:

$$J = \frac{1}{2}\int_0^9 (x_3^2 + x_6^2)dt \qquad (47)$$

subject to the following dynamic state equations

$$\dot{x}_1 = x_4 \qquad (48)$$
$$\dot{x}_2 = x_5 \qquad (49)$$
$$\dot{x}_3 = x_6 \qquad (50)$$
$$\dot{x}_4 = u_1 + 17.2656x_3 \qquad (51)$$
$$\dot{x}_5 = u_2 \qquad (52)$$
$$\dot{x}_6 = -\frac{1}{x_2}(u_1 + 27.0756x_3 + 2x_5x_6) \qquad (53)$$

The initial and the terminal state constraints are

$$x(0) = [0,22,0,0,-1,0]^T \qquad (54)$$
$$x(9) = [10,14,0,2.5,0,0]^T \qquad (55)$$

and the path inequality constraints on the states and the control are

$$|u_1(t)| \le 2.83374 \quad \forall t \in [0,9] \qquad (56)$$
$$-0.80865 \le u_2(t) \le 0.71265 \quad \forall t \in [0,9] \qquad (57)$$
$$|x_4(t)| \le 2.5 \quad \forall t \in [0,9] \qquad (58)$$
$$|x_5(t)| \le 1 \quad \forall t \in [0,9] \qquad (59)$$

Applying the iterative technique of section III and changing the time into τ , we get

For k≥ 1

Minimize

$$J^{[k]} = \frac{9}{4}\int_{-1}^{1}((x_3^{[k]})^2 + (x_6^{[k]})^2)d\tau \qquad (60)$$

subject to the following state equations

$$\dot{x}_1^{[k]} = \frac{9}{2}x_4^{[k-1]} \qquad (61)$$
$$\dot{x}_2^{[k]} = \frac{9}{2}x_5^{[k-1]} \qquad (62$$
$$\dot{x}_3^{[k]} = \frac{9}{2}x_6^{[k-1]} \qquad (63)$$

$$\dot{x}_4^{[k]} = \frac{9}{2}(u_1^{[k]} + 17.2656x_3^{[k-1]}) \qquad (64)$$
$$\dot{x}_5^{[k]} = \frac{9}{2}u_2^{[k]} \qquad (65)$$
$$\dot{x}_6^{[k]} = -\frac{9}{2}\left(\frac{1}{x_2^{[k-1]}}u_1^{[k]} + 27.0756\frac{x_3^{[k]}}{x_2^{[k-1]}} + 2\frac{x_5^{[k-1]}}{x_2^{[k-1]}}x_6^{[k]}\right) \qquad (66)$$

where

$$x^{[k]}(-1) = [0,22,0,0,-1,0]^T \qquad (67)$$
$$x^{[k]}(1) = [10,14,0,2.5,0,0]^T \qquad (68)$$

and the control inequality constraints

$$-2.83374 \le u_1^{[k]}(\tau) \le 2.83374 \quad \forall\tau \in [-1,1] \qquad (69)$$
$$-0.80865 \le u_2^{[k]}(\tau) \le 0.71265 \quad \forall\tau \in [-1,1] \qquad (70)$$

and state inequality constraints

$$-2.5 \le x_4^{[k]}(\tau) \le 2.5 \quad \forall\tau \in [-1,1] \qquad (71)$$
$$-1 \le x_5^{[k]}(\tau) \le 1 \quad \forall\tau \in [-1,1] \qquad (72)$$

Each of the problems (60)-(72) is solved by converting it into quadratic programming problem. To this end, the state variables x_1, x_2, x_3 were approximated by 9th order

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Legendre series with unknown parameters. The remaining state variables x_4, x_5, x_6 and control variables u_1, u_2 are obtained using the first five state equations. All state equations are directly satisfied except the last equation which will be replaced by $N + 1$ equality constraints by equating the coefficient of the Legendre polynomials on both sides of the equation.

Table 2 illustrates the results of simulation carried out on the crane problem using Legendre polynomials.

Table 2 Optimal values for the crane problem

Iteration i	J
0	0.005213296926173
1	0.005647797358452
2	0.005647752274546
3	0.005647708914071
4	0.005647667212744

Figures 2 to 9 show the optimal state trajectories and controls of the container crane problem using Legendre polynomials.

This problem was solved by Teo *et.al.* [19] using a piecewise constant functions to parameterize the control variables and J was found to be 0.005361. They also used a piecewise linear functions to parameterize the control variables and found $J = 0.005412$. Jaddu [5, 20] solved this problem using the second method of quasilinearization and state parameterization using Chebyshev polynomials, and J was found to be 0.00562 after three iterations. Recently, Jaddu [21] solved the same problem using iterative method with Chebyshev polynomials and obtained J = 0.005644799

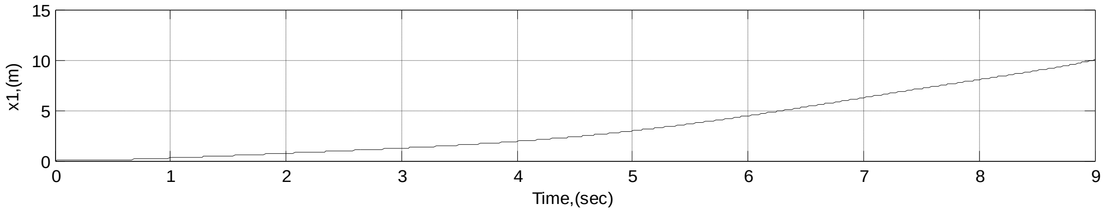

Figure 2 $x_1(t)$ optimal trajectory using Legendre Polynomials

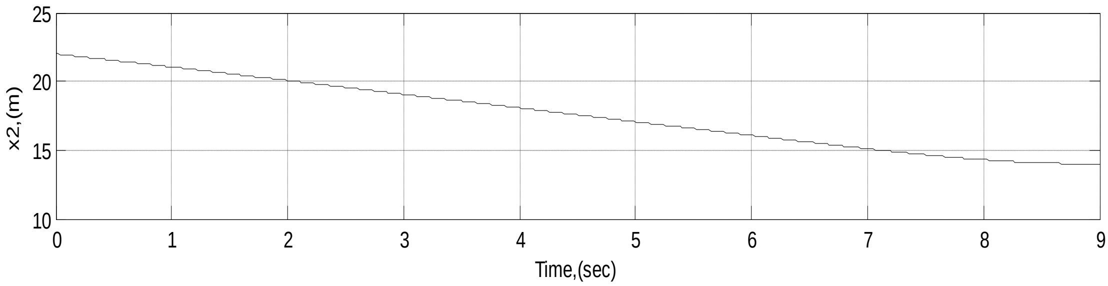

Figure 3 $x_2(t)$ optimal trajectory using Legendre Polynomials

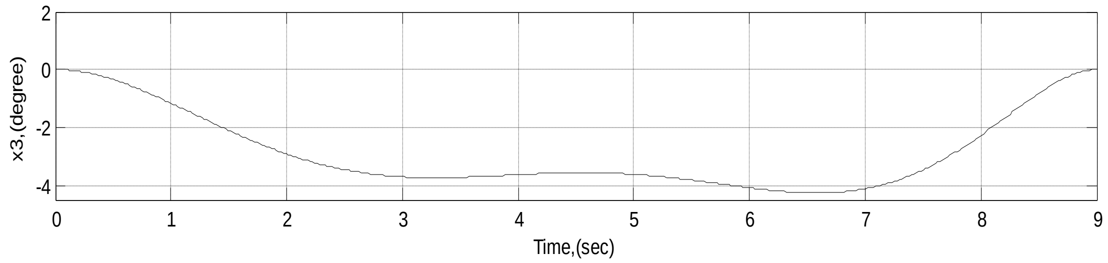

Figure 4 $x_3(t)$ optimal trajectory using Legendre Polynomials

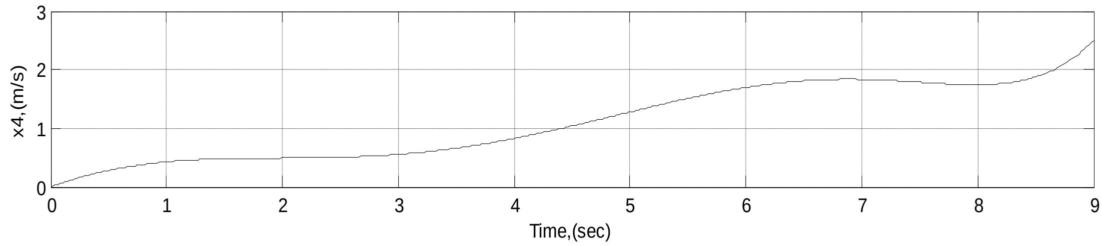

Figure 5 $x_4(t)$ optimal trajectory using Legendre Polynomials

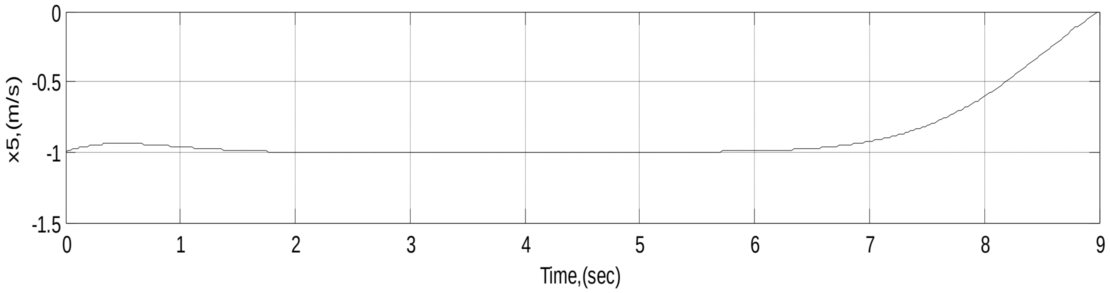

Figure 6 $x_5(t)$ **optimal trajectory using Legendre Polynomials**

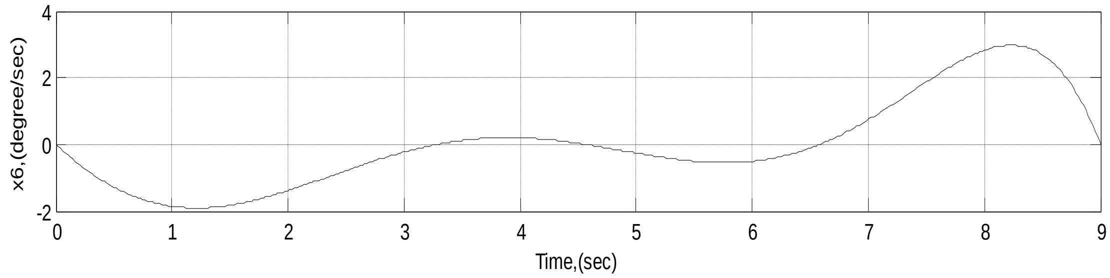

Figure 7 $x_6(t)$ **optimal trajectory using Legendre Polynomials**

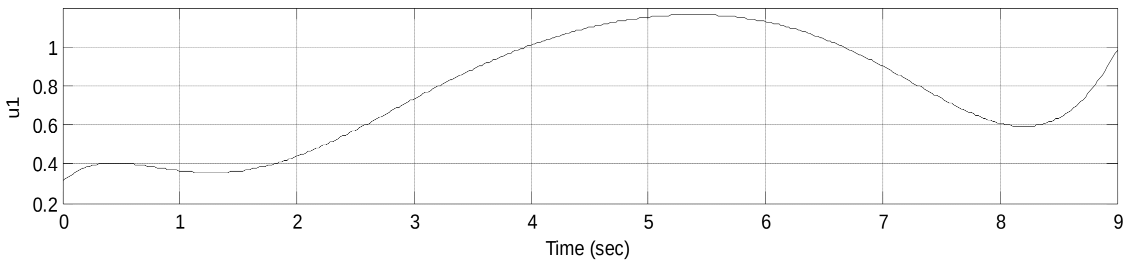

Figure 8 $u_1(t)$ **optimal control using Legendre Polynomials**

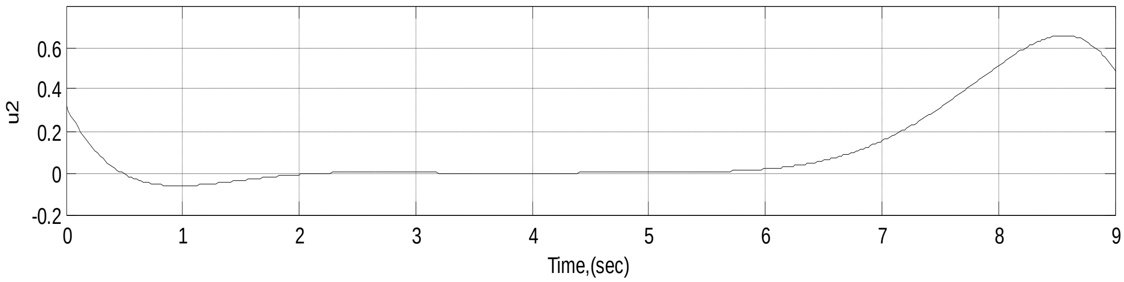

Figure 9 $u_2(t)$ **optimal control using Legendre**

VI. CONCLUSION

A recursive numerical method has been presented for solving a class of nonlinear optimal control problems with terminal state constraints and path inequality constraints on the states and the control variables.. The method is based on combining iterative technique with state parameterization using Legendre polynomials to convert the original constrained nonlinear optimal control problem into a sequence of quadratic programming problems. To show the effectiveness of the method, the simulation results of the Van der Pol oscillator problem and the container crane problem are presented and the simulation results obtained indicate that the method gives a good and comparable results with other methods.

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